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Strong approximation theorem for division algebras over $\boldsymbol{R}(X)$

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Introduction.

Let R be a Dedekind domain with the quotient field K, D be a central simple K-algebra. We call that (D, R) has the strong approximation property iff the commutator subgroup $[D^{\times}, D^{\times}]$ of $D^{\times}$ is dense in its adelization (for the precise meaning, see § 3). In this paper, when K is the rational function field $\boldsymbol{R}(X)$ of one variable over the reals, we shall prove:

SAT: (D, R) has the strong approximation property if and only if $D\otimes_{K}K_{v}$ is not a division algebra for some non-prime place v (i.e. the place v which does not come from any prime ideal of R).

If K is a global field (i.e. $[K: \boldsymbol{Q}]<\infty$ or $[K: \boldsymbol{F}_{q}(X)]<\infty$), then $[D^{\times}, D^{\times}]$ coincides with the norm 1 group $D^{(1)}$, and the result SAT of the above type is well known as SAT (Strong Approximation Theorem) of Eichler.

Swan [**11**] systematically applied SAT of Eichler to the theory of lattices over orders. Recently Hijikata [**6**], extending the scope of Swan's approach to arbitrary Dedekind domains and remarking that non-division D always has the strong approximation property, pointed out the importance of establishing SAT for a central division D over the quotient field of an arbitrary Dedekind domain.

Our result gives a first example of non-trivial SAT other than the global field. In § 1, 2, we describe the structure of the Brauer group $Br(K)$ for the algebraic function field $K=\boldsymbol{R}(X, y)$ of one variable. Although the structure of $Br(K)$ is known as "abstract groups", even for K's with much more general constant fields ([**2**], [**4**], [**5**]), we need to know some explicit isomorphism reflecting the ramification of D's. A remarkable fact is that Hasse's principle holds for $K=\boldsymbol{R}(X, y)$ (i.e. $Br(K)\rightarrow\prod Br(K_{v})$ is injective). In § 3, we formulate the strong approximation property. In § 4, we prove SAT for $R=\boldsymbol{R}[X]$. In § 5, we prove the only if part of SAT for $K=\boldsymbol{R}(X, y)$. We prove the if part of SAT for any R in $K=\boldsymbol{R}(X)$.

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