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On the number of apparent singularities of the Riemann-Hilbert problem on Riemann surface

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1. Introduction.

Let X be a compact Riemann surface of genus g. For a given finite subset S of X, let $\mathscr{D}(S)$ be the set of linear differential equations on X with meromorphic coefficients having singularities S. In this article we investigate the monodromy map from $\mathscr{D}(S)$ to the set of representations (up to conjugacy)

$$\pi_1(X-S) \longrightarrow \mathrm{GL}(n, \boldsymbol{C}).$$

The Riemann-Hilbert problem is, roughly speaking, the question whether this map is surjective or not. During this century, many mathematicians have given affirmative answers under various situations. To solve this problem, we have to think of a differential equation with some singularities, besides given S, such that around each of these singularities the monodromy is trivial. Such singularities are called *apparent* singularities. An estimate for the number of these apparent singularities was made by M. Ohtsuki [**9**] a decade ago. He used the formulation given by P. Deligne [**1**]. Deligne's formulation is explained as follows. For a holomorphic vector bundle E over X constructed by using a given representation, this problem is reduced to find a holomorphic line subbundle of E. In fact any line bundle with sufficiently small degree can be realized as a subbundle of E. To improve the estimate, however, we need a line subbundle of E with larger degree. Ohtsuki used the Riemann-Roch theorem to estimate the largest possible degree.

Recently a similar situation to Deligne's formulation is considered in a different context by P. Kronheimer and T. Mrowka [**7**]. The main tool of their argument is the Riemann-Roch-Grothendieck theorem.

In § 2 we shall use the Riemann-Roch-Grothendieck theorem for our context to improve Ohtsuki's estimate.

THEOREM A. *Let X be a compact Riemann surface of genus g and $p_1, \cdots, p_m$ distinct points in X. Assume that*

$$\rho : \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{GL}(n, \boldsymbol{C})$$