J. Math. Soc. Japan
Vol. 49, No. 1, 1997

On relativized probabilistic polynomial time algorithms

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(Received Dec. 16, 1994)

Let $\boldsymbol{SEP}_B=\{X\subseteqq\Sigma^*:\boldsymbol{P}[X]\neq\boldsymbol{BPP}[X]\}$. Bennett-Gill [**BG 81**] show that, in the Cantor space 2^{Σ^*}, $\boldsymbol{SEP}_B$ is of measure zero, and conjectured the possibility that it may be comeager. (In complexity theory there is such an example: Dowd [**Do 92**] shows that the class of m-generic oracles is of measure zero and is comeager.) We give partial answer to this possibility. Namely, we show that (i) there is a recursive oracle H such that the class $\{X:\boldsymbol{P}[X]\neq\boldsymbol{BPP}[H\oplus X]\}$ is comeager, and (ii) if we assume the existence of an oracle with an appropriate property, then the class $\boldsymbol{SEP}_B$ is comeager. These two things also hold for the class $\boldsymbol{SEP}_D=\{X:\boldsymbol{P}[X]\neq\boldsymbol{NP}[X]\cap\boldsymbol{coNP}[X]\}$. Proofs use forcing method due to Poizat [**Po 86**] with some modification. However, we do not know whether $\boldsymbol{SEP}_D$ is comeager. If $\boldsymbol{SEP}_D$ contains all generic oracles (thence it is comeager), then we would have $\boldsymbol{P}\neq\boldsymbol{NP}$, by a theorem of Blum-Impagliazzo [**BI 87**]. In the last section we state the *raison d'etre* for the above (i).

§ 1. Introduction.

For $X\subseteqq\Sigma^*$, let $\boldsymbol{C}[X]$ and $\boldsymbol{D}[X]$ be relativized complexity classes, and let $\boldsymbol{E}(\boldsymbol{C},\boldsymbol{D})=\{X:\boldsymbol{C}[X]\neq\boldsymbol{D}[X]\}$. Then, how large (or small) is $\boldsymbol{E}(\boldsymbol{C},\boldsymbol{D})$? For example, $\boldsymbol{E}(\boldsymbol{P},\boldsymbol{NP})$ has measure 1 [**BG 81**] and is comeager (e.g., [**Po 86**]), where $\boldsymbol{P}[X]$ and $\boldsymbol{NP}[X]$ are deterministic and nondeterministic polynomial time complexity classes relativized by oracle X, respectively. Now, consider the class

$$\boldsymbol{SEP}_B=\boldsymbol{E}(\boldsymbol{P},\boldsymbol{BPP})=\{X:\boldsymbol{P}[X]\neq\boldsymbol{BPP}[X]\},$$

where $\boldsymbol{BPP}[X]$ is the class of sets accepted by probabilistic polynomial time bounded oracle Turing machines with oracle X whose error probability is bounded above by some positive rationals less than 1/2. Bennett-Gill [**BG 81**] showed, among other things, that the class $\boldsymbol{SEP}_B$ has measure zero and conjectured that it may be comeager.

In this paper, we show that it is the case if $\boldsymbol{BPP}[X]$ is relativized by an appropriate oracle H. Namely, let

* This research was partially supported by Grant-in-Aid for Scientific Research (No. 06640084), Ministry of Education, Science and Culture, Japan.