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A note on the structure of the ring of symmetric Hermitian modular forms of degree 2 over the Gaussian field

Dedicated to Professor Hideo Shimizu on his sixtieth birthday

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Introduction.

In Introduction of [**14**], H. L. Resnikoff and Y.-S. Tai summarized known results about the structure of the graded ring of modular forms. They stated there as follows: Freitag [**4**] studied the Hermitian modular group of genus 2 (i.e., acting on the complex 4-dimensional Hermitian tube domain) associated with the ring $\boldsymbol{Z}[i]$ of Gaussian integers and constructed the 6 generators of the graded ring of symmetric Hermitian modular forms of even weight in terms of theta nullwerte, but the relation they satisfy is not yet known ([**14**], p. 98). The main purpose of this note is to give the explicit relation. Let $\boldsymbol{H}_2$ be the Hermitian upper half space of degree 2. The theta constant on $\boldsymbol{H}_2$ with characteristic $\boldsymbol{m}$ is defined by

$$\Theta_{\boldsymbol{m}}(Z)=\Theta(Z\,;\boldsymbol{a},\boldsymbol{b})=\sum_{\boldsymbol{g}\in M_{2\times 1}(\boldsymbol{Z}[i])}\boldsymbol{e}\left[\frac{1}{2}\left(Z\left\{\boldsymbol{g}+\frac{1+i}{2}\boldsymbol{a}\right\}+2\,\mathrm{Re}\frac{1+i}{2}{}^t\boldsymbol{b}\boldsymbol{g}\right)\right],\quad Z\in\boldsymbol{H}_2,$$

where $\boldsymbol{m}=\binom{\boldsymbol{a}}{\boldsymbol{b}}$, $\boldsymbol{a}, \boldsymbol{b}\in M_{2\times 1}(\boldsymbol{Z})$, $A\{B\}={}^t\bar{B}AB$ and $\boldsymbol{e}[s]=e^{2\pi i s}$ for $s\in\boldsymbol{C}$. Denote by $\mathcal{E}$ the set of even characteristics of degree 2 mod 2 (cf. § 1.2). Define

$$\phi_{4k}(Z):=\frac{1}{4}\sum_{\boldsymbol{m}\in\mathcal{E}}\theta_{\boldsymbol{m}}^{4k}(Z),$$

$$\chi_8(Z):=\frac{1}{3072}(\phi_4^2(Z)-\phi_8(Z)),$$

$$\chi_{10}(Z):=2^{-12}\prod_{\boldsymbol{m}\in\mathcal{E}}\theta_{\boldsymbol{m}}(Z),$$

$$\chi_{12}(Z):=2^{-15}\sum_{\text{fifteen}}(\theta_{\boldsymbol{m}_1}(Z)\cdot\theta_{\boldsymbol{m}_2}(Z)\cdots\theta_{\boldsymbol{m}_6}(Z))^2,$$

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