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On a unitary version of Suzuki's exponential product formula

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1. Let $A_1, \cdots, A_q$ be bounded skew-adjoint linear operators on a separable Hilbert space $\mathcal{H}$ over $\boldsymbol{C}$. We put the sum $A=A_1+\cdots+A_q$. For an arbitrary $x\in\boldsymbol{R}$, their exponentials $e^{xA_1}, \cdots, e^{xA_q}$ and e^{xA} are unitary. For a sequence of numbers $k_1, \cdots, k_r$ for $1\leqq k_\nu\leqq q$, we consider the product

$$Q = e^{xB_1}\cdot e^{xB_2}\cdots e^{xB_r} \tag{1.1}$$

where B_ν denotes $p_\nu A_{k_\nu}$ for some $p_\nu\in\boldsymbol{R}$. The operators Q and e^{xA} have Taylor expansions as bounded operators

$$Q = \sum_{\nu_1,\cdots,\nu_r\geqq 0}\frac{B_1^{\nu_1}\cdots B_r^{\nu_r}}{\nu_1!\cdots\nu_r!}x^{\nu_1+\cdots+\nu_r} \tag{1.2}$$

$$e^{xA} = \sum_{\nu\geqq 0}\frac{A^\nu}{\nu!}x^\nu. \tag{1.3}$$

Suppose that we can choose $p_1, \cdots, p_r$ with $\sum_{\nu=1}^r p_\nu=1$ such that

$$\|Q(x)-e^{xA}\| = O|x|^{s+1} \quad \text{for } |x|<\rho \ (\rho \text{ a positive number}) \tag{1.4}$$

for some $s\in\boldsymbol{Z}_{>0}$ ($\|\ \|$ denotes the norm of vectors in $\mathcal{H}$ or bounded operators on $\mathcal{H}$).

(1.4) is equivalent to the equality

$$\sum_{\substack{0\leqq\nu_1,\cdots,\nu_r\\ \nu_1+\cdots+\nu_r\leqq s}} x^{\nu_1+\cdots+\nu_r}\frac{B_1^{\nu_1}\cdots B_r^{\nu_r}}{\nu_1!\cdots\nu_r!} = \sum_{\nu=0}^{s}\frac{x^\nu}{\nu!}A^\nu. \tag{1.5}$$

We say then $Q(x)$ is an *s-th order approximation of* e^{xA}.

We fix the above $p_1, \cdots, p_r$ and $k_1, \cdots, k_r$. Suppose now that for each $m\in\boldsymbol{Z}_{>0}$, there exist N real numbers $p_{m,1}, \cdots, p_{m,N}$ with $\sum_{j=1}^N p_{m,j}=1$ ($N=N(m)$ depends on m) such that the ordered product

$$Q^{(m)}(x) = Q(p_{m,1}x)\cdot Q(p_{m,2}x)\cdots Q(p_{m,N}x) \tag{1.6}$$

is an m-th order approximation of e^{xA}, i.e.,