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Powers of ideals in Cohen-Macaulay rings

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1. Introduction.

Let I be an ideal in a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and assume that the field $A/\mathfrak{m}$ is infinite. For each integer $n\geqq 1$, let $I^{(n)}=\{a\in A \mid sa\in I^n$ for some $s\in A\setminus\bigcup_{p\in \mathrm{Min}_A A/I}p\}$ and call it the n-th symbolic power of I. In this paper we are going to investigate the conditions under which $I^{(n)}=I^n$ for all n. As is well-known, when I is a prime ideal and the local ring A_I is regular, $I^{(n)}=I^n$ for all $n\geqq 1$ if and only if the associated graded ring $\mathrm{G}(I)=\bigoplus_{n\geqq 0}I^n/I^{n+1}$ is an integral domain. Recalling this fact, in [**Ho**] Hochster gave a certain algorithm to check whether $\mathrm{G}(I)$ is an integral domain. Thereafter his paper has led numerous works and researches on this subject, cf. [**CN**], [**Hu1**], [**Hu2**], [**Hu4**], [**HH1**], [**HU**], [**RV**], [**SV**], [**T**]; among them we are especially interested in [**HH1**], where Huckaba and Huneke gave a criterion for the equality $I^{(n)}=I^n$ for all $n\geqq 1$ in terms of the local analytic spreads of I in the case where the analytic spread $\lambda(I)$ of I itself is relatively small. In the present paper we shall inherit the study of Huckaba and Huneke to develop their argument for the ideals of higher analytic deviation. But before going into the detail, we would like to fix some basic definitions.

We put $\lambda(I)=\dim A/\mathfrak{m}\otimes_A\mathrm{G}(I)$ and call it the analytic spread of I (cf. [**NR**]). Then we have Burch's inequalities $\mathrm{ht}_A I\leqq\lambda(I)\leqq\dim A-\inf_{n\geqq 1}\{\mathrm{depth}\, A/I^n\}$ (cf. [**Bu**]). An ideal J of A is said to be a reduction of I, if $J\subseteqq I$ and $I^{n+1}=JI^n$ for some $n\geqq 0$. For each reduction J of I we put $\mathrm{r}_J(I)=\min\{n\geqq 0 \mid I^{n+1}=JI^n\}$ and call it the reduction number of I with respect to J. A reduction J of I is said to be minimal, if it is minimal among the reductions of I. As is well-known, this is equivalent to saying that J is generated by $\lambda(I)$ elements ([**NR**]).

If $I^{(n)}=I^n$ for all $n\geqq 1$, we have $\mathrm{Ass}_A A/I^n=\mathrm{Min}_A A/I$ for all $n\geqq 1$, so that $\mathrm{depth}\, A_Q/I^nA_Q>0$ for any $Q\in\mathrm{V}(I)\setminus\mathrm{Min}_A A/I$; hence, because $\lambda(I_Q)\leqq \mathrm{ht}_A Q-\inf_{n>0}\mathrm{depth}\, A_Q/I^nA_Q$ by Burch's inequality, we have $\lambda(I_Q)<\mathrm{ht}_A Q$ for any $Q\in\mathrm{V}(I)\setminus\mathrm{Min}_A A/I$. In their paper [**HH1**] Huckaba and Huneke proved that this condition $\lambda(I_Q)<\mathrm{ht}_A Q$ for all $Q\in\mathrm{V}(I)\setminus\mathrm{Min}_A A/I$ characterizes the equality $I^{(n)}=I^n$ $(n\geqq 1)$ for a certain class of ideals I having $\lambda(I)-\mathrm{ht}_A I\leqq 2$. Following

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