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Circles on quaternionic space forms

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Introduction.

A smooth curve γ parametrized by its arc-length is called a *circle* of geodesic curvature κ ($\kappa>0$) if it satisfies the following equations with an associated unit vector field Y along γ;

$$\nabla_X X = \kappa Y\,, \qquad \nabla_X Y = -\kappa X\,,$$

where $X(t)=\dot{\gamma}(t)$. Though this definition was given by Nomizu and Yano [8] in 1974, the study on circles is just begun. We studied in [3] and [4] circles on complex space forms, and in [2] we studied them on a surface of nonpositive curvature. In this paper we study circles on a quaternion projective space and on a quaternion hyperbolic space, and show that the similar properties hold as for circles on complex space forms.

In the study of circles on complex space forms, complex torsion $\theta=\langle X, JY\rangle$, where J is the complex structure, for a circle plays an important role. On a complex projective space, every circle with $\theta=0$, ± 1 is closed, but when $0<|\theta|<1$ we have closed circles and open circles, just like geodesics on a torus. On a complex hyperbolic space, there exist a bound κ_θ for each θ such that circles with complex torsion θ are unbounded if $\kappa\leqq\kappa_\theta$ and bounded if $\kappa>\kappa_\theta$. As a corresponding invariant for circles on a quaternion Kähler manifold M, we define the *structure torsion* Θ. For a circle γ on M with associated unit vector fields X, Y, we set Θ as the norm of the projected vector $\mathrm{Proj}(X_t)$ of X_t onto the 1-dimensional quaternion subspace $\{Y_t\cdot\varepsilon\,|\,\varepsilon$ is a quaternion$\}$ (see for detail § 2). This plays the same role as the complex torsion for circles on a Kähler manifold. By using the Hopf fibration, we take a horizontal lift of a circle on a quaternionic space form. Under the identification of the algebra of quaternions with 2-dimensional complex vector space, we find it satisfies linear differential equations. By a usual method, computing eigenvalues and eigenvectors of associated matrices, we solve them and give explicit expressions of circles. With the aid of these expressions we can show some fundamental feature of them.

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