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Hessian manifolds of constant Hessian sectional curvature

By Hirohiko SHIMA

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Introduction.

Let M be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M there exists an important class of Riemannian metrics compatible with the flat affine connection D. A Riemannian metric g on M is said to be a *Hessian metric* if g is locally expressed by $g=D^2u$ where u is a local smooth function. We call such a pair (D, g) a *Hessian structure* on M and a triple (M, D, g) a *Hessian manifold* [**5**]-[**8**]. Geometry of Hessian manifolds is deeply related to Kählerian geometry and affine differential geometry. In [**1**] Amari showed that a manifold consisting of a smooth family of probability distributions admits dual affine connections and proposed geometry of statistical manifolds (i.e., manifolds with dual affine connections). The notion of a Hessian structure is the same that dual affine connections are flat. It is known that many important smooth families of probability distributions admit Hessian structures (dual flat affine connections).

In section 1 we define Hessian sectional curvatures (which correspond to holomorphic sectional curvatures for Kählerian manifolds) and study fundamental properties of spaces of constant Hessian sectional curvature. In section 2 we construct Hessian manifolds of constant Hessian sectional curvatures. We see in section 3 that certain smooth families of probability distributions are Hessian manifolds of constant Hessian sectional curvature. Chen and Ogiue [**4**] characterized Kählerian manifolds of constant holomorphic sectional curvature in terms of Chern classes. We give in section 4 a similar characterization of the spaces of constant Hessian sectional curvature by affine Chern classes. In the last section 5 we define the notion of affine Chern classes for flat affine manifolds, which correspond to Chern classes for complex manifolds.

1. Spaces of constant Hessian sectional curvature.

Let M be a Hessian manifold with Hessian structure (D, g). We express various geometric concepts for the Hessian structure (D, g) in terms of affine