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The cuspidal class number formula for the modular curves $X_1(3^m)$

Dedicated to Professor Hideo Shimizu on his 60th birthday

By Toshikazu TAKAGI

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Introduction.

In the previous papers [**7**, **8**], we determined the cuspidal class numbers of the modular curves $X_1(p^m)$ for prime numbers $p\neq 2, 3$. The purpose of this paper is to determine the cuspidal class number of the modular curve $X_1(3^m)$. Let h' be the number obtained by the substitution of 3 for p in the cuspidal class number formula for the case $p\neq 2, 3$ ([**8**, Theorem 7.1, Theorem 8.1]). Let $h_1(3^m)$ be the cuspidal class number of the curve $X_1(3^m)$. Then our main results (Theorem 3.1, Theorem 4.1) show $h_1(3^m)=h'/3$ if $m\geqq 2$. (If $m=1$, then $h_1(3)=h'/3^2=1$.) As is well known, the cuspidal divisor class groups of the modular curves are finite (Manin [**5**], Drinfeld [**1**]). As far as the author knows, the (full) cuspidal class numbers are determined in the following cases of modular curves. Let p be a prime number $\neq 2, 3$. Ogg [**6**] determined the cuspidal class number of the modular curve $X_0(p)$. Kubert-Lang [**3**, **4**] determined the cuspidal class number of the modular curve $X(p^n)$. Takagi [**7**, **8**] determined the cuspidal class number of the modular curve $X_1(p^m)$. (Klimek [**2**], Kubert-Lang [**3**, **4**] and Yu [**10**] determined the order of a certain subgroup of the cuspidal divisor class group of the modular curve $X_1(N)$.)

The contents of this paper are the following. In Section 1, we summarize some results and definitions of [**8**, Section 1-5]. In [**8**], we assumed $p\neq 2$, and the assumption $p\neq 3$ was used only in Section 6-8. So the results of this section hold for all $p\neq 2$. Here we define modified Siegel functions, construct modular units on the curve $X_1(p^m)$, embed the cuspidal divisor group into a ring R, and define a special element θ of the algebra $R\otimes\boldsymbol{Q}$. In Section 2, we determine the group of modular units on the curve $X_1(3^m)$ precisely (Theorem 2.2). In Section 3, we determine the principal divisor group as a subgroup of the ring R, which is expressed as $I_4\theta$ where I_4 is a subgroup of R. In Sections 3 and 4, we calculate the cuspidal class number of the curves $X_1(3^{2n})$ and $X_1(3^{2n+1})$, respectively (Theorem 3.1, Theorem 4.1). In the calculation, we use