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On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term

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§ 0. Introduction.

In this paper we are concerned with the decay property of the solutions to the Cauchy problem for the semilinear wave equation with a dissipative term:

$$\text{(P)}\qquad \begin{cases} u_{tt}-\Delta u+u_t+f(u)=0 & \text{in } \boldsymbol{R}^N\times[0, \infty) \\ u(x, 0)=u_0(x) \quad \text{and} \quad u_t(x, 0)=u_1(x), \end{cases}$$

where $f(u)$ is a nonlinear function like $f(u)=|u|^{\alpha}u$, $\alpha>0$.

As far as the existence and the uniqueness are concerned, the dissipative term u_t causes no difficulty and the known results for the usual nondissipative equations remain valid. That is, the problem (P) admits a weak solution $u(t)\in L^{\infty}([0, \infty); H^1\cap L^{\alpha+2})\cap W^{1,\infty}([0, \infty); L^2)$ for each $(u_0, u_1)\in H^1\cap L^{\alpha+2}\times L^2$ (cf. Strauss [**St70**]), and moreover, such a solution belongs to $C([0, \infty); H^1)\cap C^1([0, \infty); L^2)$ and is unique if $0<\alpha<4/(N-2)$ ($0<\alpha<\infty$ if $N=1, 2$) (cf. Ginibre and Velo [**GV85**], Brenner [**Br89**] etc.). Further, roughly speaking, if $0<\alpha<4/(N-2)$ and (u_0, u_1) are smooth, the solution u is also smooth. (Cf. Jörgens [**Jö61**], Pecher [**Pe76**], Brenner and W. v. Wahl [**BW81**] etc.. See also Grillakis [**Gr90**] for the case $N=3$, $\alpha=4$.)

The purpose of this paper is to derive certain decay rates in several energy norms of the solutions of the problem (P) by use of the effect of the dissipative term u_t. When $0<\alpha\leqq 2/(N-2)$, we can employ a rather standard energy method to show, for example, in the typical case $N=3$ and $\alpha=2$,

$$\sum_{i=0}^{m+1}\|D_t^i D_x^{m+1-i}u(t)\| \leqq c_{m+1}(1+t)^{-(m+1)/2},$$

where c_{m+1} is some positive constant depending on $\|u_0\|_{H^{m+1}}+\|u_1\|_{H^m}$.

For the case $2/(N-2)<\alpha<4/(N-2)$, however, our problem is more delicate and we must utilize precise L^p-L^q-estimates for the linear equation. In this