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Ricci curvature, diameter and optimal volume bound

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§ 1. Motivation and main results.

It is well known that a complete Riemannian n-manifold M with the Ricci curvature $Ric(M) \geqq (n-1)k$ and the diameter $d(M) \leqq D$ has the volume bounded above by the volume $\tilde{v}_k(D)$ of a D-ball in the simply connected space form M_k^n with the constant sectional curvature k. In other words, if we rescale and normalize the metric so that $d(M)=\pi$ and consider the class $\boldsymbol{M}_k$ of all closed Riemannian n-manifold with $Ric(M) \geqq (n-1)k$ and $d(M)=\pi$, then the volume defines a function on $\boldsymbol{M}_k$:

$$vol: \boldsymbol{M}_k \longrightarrow \boldsymbol{R}^+$$

with the range in the interval $(0, \tilde{v}_k(\pi)]$. Note that the Myers theorem implies that k must be smaller than or equal to 1 since $d(M)=\pi$.

For $k=1$, the maximal diameter sphere theorem of Cheng [**Ch**] implies that $\boldsymbol{M}_k$ contains only one element, the n-sphere with its canonical metric *can*. Hence the range of *vol* on $\boldsymbol{M}_1$ contains the single value $\tilde{v}_1(\pi)$. To see that there is no positive lower bound on the function *vol* defineded on $\boldsymbol{M}_k$ for $k \leqq 0$, one can consider the flat tori: $S^1(\varepsilon) \times T^{n-1}$, $\varepsilon > 0$ where $S^1(\varepsilon)$ is the circle with radius ε in $\boldsymbol{R}^2$ and T^{n-1} is a flat $(n-1)$-torus. For positive $k<1$, one can consider the suspension M_ε of an $(n-1)$-sphere, S_ε^{n-1}, in $\boldsymbol{R}^n$ with radius $\varepsilon<1$. Namely, $M_\varepsilon = S_\varepsilon^{n-1} \times_{\sin}[0, \pi]$. Note that M_ε is the n-sphere S^n. Then smooth the two singular points and rescale the metric to obtain a metric g_ε on M_ε with $d(g_\varepsilon) = \pi$ and $\min Ric(g_\varepsilon) \geqq 1-\eta_1(\varepsilon)$ and $vol(g_\varepsilon) \leqq \eta_2(\varepsilon)$ where the positive functions $\eta_1(\varepsilon)$ and $\eta_2(\varepsilon)$ approach zero as ε goes to zero. See also [**GP**] for a similar construction. This indicates that the lower bound of *vol* on $\boldsymbol{M}_k$ is also zero for positive $k<1$.

For the upper bound of *vol* on $\boldsymbol{M}_k$, one may ask if the upper bound $\tilde{v}_k(\pi)$ is obtainable by some Riemannian n-manifold in $\boldsymbol{M}_k$? The answer is yes only when $k=1/4$ or 1. They are obtained by $(\boldsymbol{RP}^n, 4can)$ and (S^n, can), respectively. Therefore it is natural to ask the following

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