J. Math. Soc. Japan
Vol. 47, No. 1, 1995

Braid relations, meta-abelianizations and the symbols $\{p, -1\}$ in $K_2(2, \boldsymbol{Z}[1/p])$

By Jun MORITA

(Received Oct. 3, 1991)
(Revised June 10, 1993)

1. Introduction.

In [5], a finite presentation of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ was given for a prime number p, where two braid relations appeared. The purpose of this paper is to study the group structures of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ and $K_2(2, \boldsymbol{Z}[1/p])$. To do this, first we will look at the braid groups (cf. [1]). Then we obtain the following.

$$\text{(1)} \qquad \mathrm{St}\left(2, \boldsymbol{Z}\left[\frac{1}{p}\right]\right)^{mab} \cong \begin{cases} Z_3 \ltimes (Z_2 \times Z_2) & \text{if } p=2; \\ Z_8 \ltimes Z_3 & \text{if } p=3; \\ Z_{p^2-1} \ltimes (Z \times Z) & \text{otherwise.} \end{cases}$$

(2) If $p \neq 2, 3, 5, 11$, then $\{p, -1\}^2 \neq 1$ in $K_2(2, \boldsymbol{Z}[1/p])$.

(3) If $p \neq 2, 3, 5, 11$, then $K_2(2, \boldsymbol{Z}[1/p]) \not\cong Z \times Z_{p-1}$.

It is already known that

$$(*) \qquad K_2(2, \boldsymbol{Z}_S) \cong Z \times \prod_{p \in S} Z_{p-1}$$

if S is one of $\{$*the first n successive prime numbers*$\}$ with $n \geqq 1$, $\{3\}$, $\{2, 5\}$, $\{2, 3, 7\}$, $\{2, 3, 11\}$, $\{2, 3, 5, 11\}$, $\{2, 3, 13\}$, $\{2, 3, 7, 13\}$, $\{2, 3, 17\}$, and $\{2, 3, 5, 19\}$, where $\boldsymbol{Z}_S = \boldsymbol{Z}[1/p]_{p \in S}$ (cf. [4], [5]). One might expect that (*) holds for every set S of prime numbers. But the above (3) tells us that (*) is not true in general.

Here, we fix our notation as follows. Let $\boldsymbol{Z}$ be the ring of rational integers, $\boldsymbol{Q}$ the field of rational numbers, and $\boldsymbol{R}$ the field of real numbers. For elements x, y in a group, we set $x^y = yxy^{-1}$. For subgroups H_1, H_2 of a group G we denote by $[H_1, H_2]$ the subgroup of G generated by $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ for all $h_1 \in H_1$, $h_2 \in H_2$. Then, put $G' = [G, G]$, $G'' = [G', G']$, $G^{ab} = G/G'$, $G^{mab} = G/G''$ and $G'^{ab} = G'/G''$. We use Z_m for the cyclic group of order m, and Z for Z_∞. If a group H acts on another group K, the semi-direct product of H and K is denoted by $H \ltimes K$. And $\langle$*generators*$|$*relations*$\rangle$ means a group presentation as usual.