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A theorem of Hardy-Littlewood for harmonic functions satisfying Hölder's condition

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1. Introduction.

Our aim in this paper is to give an extension of a result of Hardy-Littlewood [**2**, Theorems 40 and 41] for holomorphic functions on the unit disc.

Let $B(x, r)$ denote the open ball centered at x with radius r. We denote by $\boldsymbol{B}$ the unit ball $B(0, 1)$ of R^n, and by $d(x)$ the distance of x from the boundary $\partial\boldsymbol{B}$, that is, $d(x)=1-|x|$.

An easy modification of the proof of [**1**, Theorem 5.1] deduces the following results (see also [**3**, Theorem 15.8]).

THEOREM A. *Let u be a harmonic function on $\boldsymbol{B}$ and $0<\alpha\leqq 1$. Then u satisfies*

$$|\nabla u(x)| \leqq Md(x)^{\alpha-1} \qquad \text{for any } x\in\boldsymbol{B}$$

if and only if

$$|u(x)-u(y)| \leqq M|x-y|^{\alpha} \qquad \text{for any } x\in\boldsymbol{B} \text{ and } y\in\boldsymbol{B}, \tag{1}$$

where ∇ denotes the gradient.

If u satisfies (1), then we say that u satisfies Hölder's condition of exponent α in $\boldsymbol{B}$.

In this paper let M denote various constants, whose value may change from one occurrence to the next.

THEOREM B. *Let u be a harmonic function on $\boldsymbol{B}$. Then u satisfies*

$$|\nabla u(x)| \leqq Md(x)^{-1} \qquad \text{for any } x\in\boldsymbol{B}$$

if and only if $u\in BMO(\boldsymbol{B})$, that is,

$$\frac{1}{|B|}\int_B \left| u(y)-\frac{1}{|B|}\int_B u(z)dz \right| dy \leqq M$$

for any open ball $B=B(x, r)\subseteqq\boldsymbol{B}$.