J. Math. Soc. Japan
Vol. 47, No. 1, 1995

Spin^q structures

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(Received Oct. 6, 1992)
(Revised May 31, 1993)

Introduction.

In this paper the notion of Spin^q-structure is introduced and some of the basic materials related to it will be discussed.

To explain the motivation briefly, let us take an n-dimensional compact oriented Riemannian manifold X. The reduced structure group $SO(n)$ $(n \geqq 3)$ has the universal covering group $\mathrm{Spin}(n)$ called the Spin group, together with the short exact sequence

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\xi_0} SO(n) \longrightarrow 1 .$$

A principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}(n)}$ with a $\mathrm{Spin}(n)$-equivariant bundle map ξ_0 from $P_{\mathrm{Spin}(n)}$ to the reduced structure bundle $P_{SO(n)}$ is then called a *Spin-structure* on X ([**3**, §5]). As is well-known, it plays a role of great importance particularly in the study of the interrelations between topology, geometry and analysis. However, to our regret, it turns out apparently not always to be effective for researching into a complex manifold X, $w_2(X) \equiv c_1(X)$ (mod 2), because there exists a Spin-structure on X if and only if the second Stiefel-Whitney class vanishes, $w_2(X)=0$. To avoid this disadvantage, the notion of Spin^c-structure was introduced ([**3**, §5 Remark 4]). That is, using the unitary group $U(1)$ $(=SO(2))$, the Spin group is twisted into the Spin^c group, $\mathrm{Spin}^c(n) \equiv \mathrm{Spin}(n) \times_{\boldsymbol{Z}_2} U(1)$, together with the short exact sequence

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\xi} SO(n) \times U(1) \longrightarrow 1 ,$$

where $\xi([\varphi, z]) = (\xi_0(\varphi), z^2)$. The *Spin^c-structure* is then defined to be a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c(n)}$ with a $\mathrm{Spin}^c(n)$-equivariant bundle map $\xi : P_{\mathrm{Spin}^c(n)} \to P_{SO(n)} \times P_{U(1)}$, where $P_{U(1)}$ is a certain principal $U(1)$-bundle. Since the existence can be characterized by the condition that $w_2(X)$ is the mod 2 reduction of an integral class, a complex structure certainly induces a Spin^c-structure. The study of complex manifolds using this structure is also too vast to survey here.

Let us consider next the case where X has an almost quaternionic structure. The so-called quaternionic Kähler manifolds are examples. The research in