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Multiple stochastic integrals appearing in the stochastic Taylor expansions

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Introduction.

Let $V_0, V_1, \cdots, V_n$ be smooth vector fields on $\boldsymbol{R}^d$ (in general may be a smooth manifold) and we consider the stochastic differential equation (abbr. SDE) on $\boldsymbol{R}^d$:

$$\begin{aligned} dX_t &= \sum_{i=0}^{n} V_i(X_t) \circ dB_t^i \\ X_0 &= x \end{aligned} \tag{0.1}$$

where $(B_t^1, \cdots, B_t^n)$ is the n-dimensional Brownian motion starting from $0 \in \boldsymbol{R}^d$, $B_t^0 = t$ and the symbol $\circ$ denotes the Stratonovich integral. Let us denote by $X(t, x)$ the solution to this SDE. (Here let us suppose some appropriate condition under which the SDE (0.1) has a unique and global solution.) Then as $t \downarrow 0$, $X(t, x)$ is expanded as follows:

$$X(t, x) \sim x + \sum_{m=1}^{\infty} \sum_{i_1, \cdots, i_m=0}^{n} B_t^{i_1 \cdots i_m} V_{i_1} \cdots V_{i_m}(x). \tag{0.2}$$

This is called the stochastic Taylor expansion and has a sense as an asymptotic expansion, and generally does not converge in probability for given $t>0$. In the expansion, $B_t^{i_1 \cdots i_m}$ is a multiple stochastic integral for $B_t^{i_1}, \cdots, B_t^{i_m}$ defined by

$$B_t^{i_1 \cdots i_m} = \int_0^t \circ dB_{s_m}^{i_m} \int_0^{s_m} \circ dB_{s_{m-1}}^{i_{m-1}} \cdots \circ dB_{s_2}^{i_2} \int_0^{s_2} \circ dB_{s_1}^{i_1}. \tag{0.3}$$

When we study the asymptotic problem of quantity relative to $X(t, x)$ such as heat kernel, the expansion (0.2) is basic and there is a routine as follows: We decompose $X(t, x)$ as $X(t, x) = F(t, x) + R(t, x)$ such that $F(t, x)$ is a finite expansion in (0.2) cut in the m_0-th term (m_0 is chosen large enough in advance) and $R(t, x)$ is the remainder, and then show $R(t, x)$ to be actually negligible in an appropriate sense and hence reduce the problem to that for $F(t, x)$, i.e., a finite system $B_t^{i_1 \cdots i_m}$, $0 \leqq i_1, \cdots, i_m \leqq n$, $1 \leqq m \leqq m_0$.

In this paper, we are interested in an infinite system $B_t^{i_1 \cdots i_m}$, $0 \leqq i_1, \cdots, i_m \leqq$