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Spectra and geodesic flows on nilmanifolds: Reductions of Hamiltonian systems and differential operators

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0. Introduction.

Two compact Riemannian manifolds are said to be isospectral if their associated Laplace-Beltrami operators have the same spectra. In 1964 J. Milnor [**15**] first gave an example of a pair of isospectral 16-dimensional flat tori which are not isometric to each other. Later, other examples of pairs of isospectral but not isometric Riemannian manifolds were given by several authors. (See, e. g., the review paper [**2**].) In particular, T. Sunada [**17**] gave a general technique for constructing pairs of isospectral manifolds with a common finite Riemannian covering. In 1984 C. S. Gordon and E. Wilson [**9**] exhibited for the first time continuous one-parameter families of nonisometric, isospectral metrics (isospectral deformations of metrics). Their examples are constructed on solvmanifolds or nilmanifolds, i. e., manifolds whose universal Riemannian coverings are solvable or nilpotent Lie groups with left-invariant metrics. Their idea was further developed, involving Sunada's method, in the subsequent articles [**4**], [**5**].

In this paper we analyze the isospectral deformations on nilmanifolds constructed by Gordon, Wilson and DeTurck from the viewpoint of dynamical systems: classical Hamiltonian systems (geodesic flows) and quantized systems (Laplace-Beltrami operators). A key to our consideration is the reduction procedure of (classical) Hamiltonian systems formulated by Marsden and Weinstein [**14**] and its analog for differential operators (quantum systems). We decompose the system of geodesic flow and its quantization (the Laplace-Beltrami operator) into families of classical and quantum reduced systems, respectively. Another key point is the notion of (pseudo-)restricted-inner transformations of a Lie algebra, which is motivated by the notion of almost-inner derivations by Gordon and Wilson [**9**]. We make clear what occurs in the reduced systems under the deformations of Riemannian metrics generated by a (pseudo-)restricted-

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