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Group actions and deformations for harmonic maps

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Introduction.

From the theory of integrable systems it is known that harmonic maps from a Riemann surface to a Lie group may be studied by infinite dimensional methods (cf. [**ZM**], [**ZS**]). This was clarified considerably by the papers [**Uh**], [**Se**], especially in the case of maps from the Riemann sphere S^2 to the unitary group U_n. The basic connection with infinite dimensional methods is the correspondence between harmonic maps $S^2 \to G$ and "extended solutions" $S^2 \to \Omega G$, where G is any compact Lie group and ΩG is its (based) loop group. In [**Uh**] this was used in two ways (in the case $G=U_n$):

(1) to introduce a *group action* of matrix valued rational functions on harmonic maps, and

(2) to prove a *factorization theorem* for harmonic maps, which unifies and extends many of the known results on the classification of harmonic maps from S^2 into various homogeneous spaces.

In [**Se**] it was shown that the factorization theorem can be proved very naturally by using the "Grassmannian model" of ΩG, which is an identification of ΩG with a certain infinite dimensional Grassmannian (see [**PS**]). In this paper we shall show how the group action may be interpreted in terms of the Grassmannian model. The advantages of this point of view are that the geometrical nature of the action is emphasized, and that calculations become easier. We shall illustrate this by giving some applications to deformations of harmonic maps. By using some elementary ideas from Morse theory, we obtain new results on the connectedness of spaces of harmonic maps, a subject which has been studied recently by various *ad hoc* methods (for example, in [**Ve1**], [**Ve2**], [**Ve3**], [**Lo**], [**Kt**]).

The paper is arranged as follows. In §1 we give the basic definitions, including that of a "generalized Birkhoff pseudo-action". The latter is an action of k-tuples of loops γ on extended solutions Φ, denoted by $(\gamma, \Phi) \mapsto \gamma^{\#}\Phi$. This definition involves a Riemann-Hilbert factorization (a generalization of the Birkhoff factorization for loops), and is an example of a "dressing action" in the theory of integrable systems. Because the factorization cannot always be carried out, the action is defined only for certain γ and Φ, so we call it a pseudo-action.