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Borsuk-Ulam theorem and Stiefel manifolds

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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Introduction.

There are several different, but equivalent versions of the classical Borsuk-Ulam theorem. One of them can be stated as follows:

THE CLASSICAL BORSUK-ULAM THEOREM. *Let* S^n *be the unit sphere in euclidean* $(n+1)$*-space* $\boldsymbol{R}^{n+1}$. *If* $f: S^n \to \boldsymbol{R}^n$ *is a* $\boldsymbol{Z}_2$*-map, i.e., satisfies* $f(-x)=-f(x)$ *for all* $x \in S^n$, *then* $f^{-1}(0)$ *is nonempty.*

Many authors have been contributing to generalizing and extending the Borsuk-Ulam theorem in various ways (see Steinlein [10]). Recently E. Fadell-S. Husseini and J. W. Jaworowski independently introduced an *ideal-valued cohomological index theory* and extended the theorem to maps of Stiefel manifolds, see [2], [3], [4] and [5].

Let $(\boldsymbol{R}^n)^k$ denote the cartesian product of k copies of $\boldsymbol{R}^n$. Any point of $(\boldsymbol{R}^n)^k$ is represented by a $(k \times n)$-matrix. Then the k-th orthogonal group $O(k)$ acts on $(\boldsymbol{R}^n)^k$ by matrix multiplication on the left. When $k \leqq n$, the Stiefel manifold $V_k(\boldsymbol{R}^n)$ of orthonormal k-frames in $\boldsymbol{R}^n$ can be considered a subspace of $(\boldsymbol{R}^n)^k$ on which $O(k)$ acts freely. In [2], [3], Fadell and Husseini considered $\boldsymbol{Z}_2^k$-maps $f: V_k(\boldsymbol{R}^n) \to (\boldsymbol{R}^{n-k})^k$ where $\boldsymbol{Z}_2^k = \boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2$ (k times) is a subgroup of $O(k)$ which is diagonally imbedded, and they estimated the cohomological size of $f^{-1}(O)/\boldsymbol{Z}_2^k$ where O is the zero of $(\boldsymbol{R}^{n-k})^k$. In [4], [5], Jaworowski considered $O(2)$-maps $f: V_2(\boldsymbol{R}^n) \to (\boldsymbol{R}^l)^2$ and estimated the cohomological size of $f^{-1}(T)/O(2)$, where $T = \{A \in (\boldsymbol{R}^l)^2 \mid \operatorname{rank} A < 2\}$.

In the present paper we will consider more general class of maps of Stiefel manifolds and generalize their results. We will employ (mod 2) cup_1*-length*, denoted $\mathrm{cup}_1(X)$, as a measure of the cohomological size of a space X. $\mathrm{cup}_1(X)$ is defined to be the greatest number s such that there exist $x_1, \cdots, x_s \in H^1(X; \boldsymbol{Z}_2)$ with $x_1 \cup \cdots \cup x_s \neq 0$. The inequality $\mathrm{cup}_1(X) \geqq 0$ means X is at least nonempty. When $x_1, \cdots, x_s$ can be taken in any positive degrees, the usual *cup-length*, denoted $\mathrm{cup}(X)$, is defined. Then $\mathrm{cup}_1(X) \leqq \mathrm{cup}(X) < \mathrm{cat}(X)$, where