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The adjoint action of a Lie group on the space of loops

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1. Introduction.

Let G be a compact, connected, simply connected Lie group and e its unit. Denote by ΛG the space of free loops on G and by ΩG the space of based loops on G the base point e. By the multiplication of G and compact open topology ΛG is a topological group and ΩG is a closed normal subgroup. There is an extension of groups

$$1 \longrightarrow \Omega G \xrightarrow{i} \Lambda G \xrightarrow{\pi} G \longrightarrow 1$$

with a canonical section $s: G \to \Lambda G$ defined by $s(g)(t)=g$ for any $g \in G$ and $t \in [0, 1]$. We denote the multiplications of G and ΛG by μ and λ respectively and the multiplication of ΩG by the same symbol λ. We also define maps $\mathrm{Ad}: G \times \Omega G \to \Omega G$ by $\mathrm{Ad}(g, l)(t)=gl(t)g^{-1}$ for $g \in G$, $l \in \Omega G$ and $t \in [0, 1]$ and $\Phi: \Omega G \times G \to \Lambda G$ by $\Phi(l, g)=\lambda(l, s(g))$. Then Φ is a homeomorphism and the diagram

$$\text{(1.1)} \quad \begin{array}{ccccc} \Omega G \times G \times \Omega G \times G & \xrightarrow{\omega} & \Omega G \times \Omega G \times G \times G & \xrightarrow{\lambda \times \mu} & \Omega G \times G \\ \Big\downarrow{\scriptstyle \Phi \times \Phi} & & & & \Big\downarrow{\scriptstyle \Phi} \\ \Lambda G \times \Lambda G & & \xrightarrow{\quad\lambda\quad} & & \Lambda G \end{array}$$

is commutative where ω is the composition

$$(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G \times G} \times \mathrm{Ad} \times 1_G) \circ (1_{\Omega G} \times \Delta_G \times 1_{\Omega G \times G}).$$

The purpose of this paper is to show the following:

THEOREM 1. *Let G be a compact, connected, simply connected Lie group and p a prime. Then the following three conditions are equivalent:*

(1) *$H^*(G; \boldsymbol{Z})$ is p-torsion free,*

(2) *$H^*(\mathrm{Ad}; \boldsymbol{Z}/p)=H^*(p_2; \boldsymbol{Z}/p)$, where p_2 is the second projection,*

(3) *$H^*(B\Lambda G; \boldsymbol{Z}/p)$ is isomorphic to $H^*(BG; \boldsymbol{Z}/p) \otimes H^*(G; \boldsymbol{Z}/p)$ as an algebra.*