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Elliptic differential inequalities with applications to harmonic maps

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Introduction.

Harmonic maps $\phi:(M, g)\rightarrow(N, h)$ between Riemannian manifolds are the smooth critical points of the *energy functional*

$$E(\phi)=\int_{M}e(\phi)dV_{M}\ ,$$

where $e(\phi)=(1/2)|d\phi|^{2}$ is the *energy density* of ϕ. Or, equivalently, the C^{2} solutions of the elliptic system

$$\mathrm{Trace}_{g}\nabla d\phi=0\ . \tag{0.1}$$

The left-hand side of (0.1) is the *tension field* of ϕ, denoted $\tau(\phi)$; it is a vector field along ϕ: we refer to the surveys [5], [6] for complete definitions and background.

Since the pioneering work of Eells and Sampson ([7] (1964)), harmonic maps have attracted the interest of both geometers and analysts: during the early stages of the theory, research was focused on maps between compact manifolds. Indeed, in a compact setting a harmonic map provides a strong candidate for a "best map" in a prescribed homotopy class; and a natural generalization of the concept of closed geodesic.

More recently, harmonic maps of non-compact domains have become object of growing interest: as a significant example, we quote the discovery of a new family of harmonic maps $\phi:\boldsymbol{R}^{2}\rightarrow\boldsymbol{H}^{2}$ of rank two almost everywhere; that was obtained by Choi and Treibergs [4], using a version of Ruh-Vilms' Theorem for constant mean curvature hypersurfaces of Minkowski 3-space. It is natural to view the study of harmonic maps of non-compact domains as a generalization of the theory of harmonic functions $f:M\rightarrow\boldsymbol{R}$ on complete Riemannian manifolds [18]; however, we point out two key differences:

a) a single equation — i.e., $\Delta f=0$ — is replaced by a system — i.e., (0.1).

b) the curvature of the range plays a role, making system (0. 1) non-linear.