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On cohomology groups attached to towers of algebraic curves

To Professor Shoshichi Kobayashi on his 60th birthday

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Introduction.

Let $\boldsymbol{Q}^*$ be a subfield of $\bar{\boldsymbol{Q}}$, and suppose that we are given an open immersion $j^*: Y^* \hookrightarrow X^*$ of smooth and geometrically irreducible algebraic curves over $\boldsymbol{Q}^*$, where Y^* (resp. X^*) is affine (resp. proper) over $\boldsymbol{Q}^*$. Let $j: Y \hookrightarrow X$ be the base change of j^* to $\bar{\boldsymbol{Q}}$. Then for any abelian (or $\boldsymbol{Z}_l$-) sheaf F on the étale site of Y^*, we have three kinds of étale cohomology groups: $H^1(Y, F)$, $H_c^1(Y, F) = H^1(X, j_!F)$ and $H_P^1(Y, F) := H^1(X, j_*F) \cong \mathrm{Im}\,(H_c^1(Y, F) \to H^1(Y, F))$. Such cohomology groups, being equipped with the action of the Galois group $G_{\boldsymbol{Q}^*} := \mathrm{Gal}\,(\bar{\boldsymbol{Q}}/\boldsymbol{Q}^*)$, often come up as interesting objects when Y^*, X^* and F are suitably chosen. For instance, they naturally appear in the study of the elliptic modular forms, when Y^* and X^* are the canonical models of the modular curves (cf. [**D**]).

The purpose of this paper is to study the cohomology groups of the same type, not for a single pair $Y^* \hookrightarrow X^*$, but for a tower of algebraic curves. Namely, let $Y^* \hookrightarrow X^*$ be as above, and consider a tower $\{Y_n^*\}_{n \in N}$ of geometrically irreducible algebraic curves over $\boldsymbol{Q}^*$, all of which are étale coverings of Y^*. In the text, this tower will be subject to some simple "axioms", which include that all $Y_n := Y_n^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}}$ are Galois coverings of Y, and that $\mathfrak{G} := \varprojlim_{n \in N} \mathrm{Gal}(Y_n/Y)$ is an "almost pro-l group" with a prime number l. (See § 1 for details, where two basic examples of such towers can be also found.) Let X_n^* be the normalization of X^* in Y_n^*, and put $X_n := X_n^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}}$. The group $\mathfrak{G}$ naturally acts on the various cohomology groups $H^1(Y_n, \boldsymbol{Z}_l)$, $H_c^1(Y_n, \boldsymbol{Z}_l)$ and $H_P^1(Y_n, \boldsymbol{Z}_l) \cong H^1(X_n, \boldsymbol{Z}_l)$; and hence we may consider them as modules over the completed group algebra $\mathcal{A} := \boldsymbol{Z}_l[[\mathfrak{G}]]$, as well as $G_{\boldsymbol{Q}^*}$-modules. Now we would like to put such cohomology groups together to get *single* cohomology theories corresponding to "H^1", "H_c^1" and "H_P^1", respectively attached to the given tower.

Natural candidates for such cohomology theories are simply the projective limits $\varprojlim_{n \in N} H^1(Y_n, \boldsymbol{Z}_l)$, $\varprojlim_{n \in N} H_c^1(Y_n, \boldsymbol{Z}_l)$ and $\varprojlim_{n \in N} H^1(X_n, \boldsymbol{Z}_l)$ relative to the trace mappings; and our aim is to study their structure. Let Z be the maximum connected pro-l étale Galois covering of the scheme $\varprojlim_{n \in N} Y_n$, and let $\mathfrak{F}$