J. Math. Soc. Japan
Vol. 45, No. 1, 1993

The set of vector fields with transverse foliations

By Plácido ANDRADE

(Received Sept. 5, 1991)
(Revised Dec. 4, 1991)

Introduction.

There exist precise criterions to decide whether a given C^1 flow ϕ on an m dimensional closed manifold M admits a cross-section. For example, one has the asymptotic cycles [**Sc**] as well as homology directions [**Fr**]. Both of these make use of the first real homology group of M. On the other hand, there does not exist a general criterion to decide whether a flow admits a transverse foliation. However, in the case of a three manifold this problem is solved for certain types of flows, like flows whose orbits are compact [**Mi**], [**Wo**], [**E-H-N**], Morse-Smale flows [**Go1**] and Smale flows [**Go2**]. In this paper we treat the problem of extending the result of Goodman's criterion to a general vector field on a three manifold. We found that the natural extension should be in terms of what we call "homotopy direction" [**An2**]. Using this notion we define the set $\mathcal{L}(M)$ of vector fields whose flows are homotopically linked (§ 2). Although we were not completely successful, we obtained unexpected properties which are described in the theorems below.

Let M be a smooth three dimensional closed manifold. We assume that M is oriented and for convenience we shall fix a Riemannian metric. Every flow ϕ appearing henceforth is generated by a vector field $\dot{\phi}$ in $NSX(M)$, the space of C^1 non-singular vector fields on M endowed with the C^0 topology and every foliation $\mathcal{F}$ is a codimension one transversely oriented foliation on M given by a C^1 coordinate systems. We denote by $\pitchfork(M)$ the topological subspace of $NSX(M)$ of vector fields whose flows admit a transverse foliation and by $\pitchfork(\overline{M})$ its closure.

0.1 THEOREM. *The sets $\pitchfork(M)$ and $\mathcal{L}(M)$ are open and not dense in $NSX(M)$ and satisfy the inclusions*

$$\pitchfork(M) \subset \mathcal{L}(M) \subset \pitchfork(\overline{M}) .$$

We construct a flow to show the following

Partly supported by S. C. T/Pr (Brazil).