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Mappings of moduli spaces for harmonic eigenmaps and minimal immersions between spheres

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1. Introduction and preliminaries.

Let $\mathcal{H}^p=\mathcal{H}^p_{S^m}$ denote the vector space of spherical harmonics of order $p\geqq 1$ on the Euclidean m-sphere S^m, $m\geqq 2$. We think of a spherical harmonic as a degree p homogeneous harmonic polynomial in $m+1$ variables or as an eigenfunction of the Laplace-Beltrami operator Δ^{S^m} with eigenvalue $\lambda_p=p(p+m-1)$ (obtained from the polynomial by restriction to S^m). A map $f: S^m\to S_V$ into the unit sphere of a Euclidean vector space V is said to be a *λ_p-eigenmap* if all components of f belong to $\mathcal{H}^p$, i.e., for $\mu\in V^*$, we have $\mu\circ f\in\mathcal{H}^p$. (Note that a λ_p-eigenmap is nothing but a harmonic map with energy density $\lambda_p/2$ [2].) $f: S^m\to S_V$ is *full* if the image of f in V spans V. In general, restricting to *span im* $f\cap S_V$, f gives rise to a full λ_p-eigenmap that we will denote by the same symbol. Two λ_p-eigenmaps $f_1: S^m\to S_{V_1}$ and $f_2: S^m\to S_{V_2}$ are *equivalent* if there exists an isometry $U: V_1\to V_2$ such that $f_2=U\circ f_1$.

The universal example of a λ_p-eigenmap is given by the standard minimal immersion $f_{\lambda_p}: S^m\to S_{\mathcal{H}^p}$ defined by

$$f_{\lambda_p}(x)=\sum_{j=0}^{n(\lambda_p)} f^j_{\lambda_p}(x)f^j_{\lambda_p},$$

where $\{f^j_{\lambda_p}\}_{j=0}^{n(\lambda_p)}\subset\mathcal{H}^p$ is an orthonormal basis with respect to the normalized L_2-scalar product

$$\langle h, h'\rangle_p=\frac{n(\lambda_p)+1}{vol(S^m)}\int_{S^m} hh'v_{S^m}. \tag{1}$$

Here v_{S^m} is the volume form on S^m, $vol(S^m)=\int_{S^m}v_{S^m}$ is the volume of S^m and

$$n(\lambda_p)+1=\dim\mathcal{H}^p=(2p+m-1)\frac{(p+m-1)!}{(p+1)!(m-1)!}. \tag{2}$$

f_{λ_p} is clearly full and does not depend on the orthonormal basis.

f_{λ_p} is universal in the sense that, for any λ_p-eigenmap $f: S^m\to S_V$, there exists a linear map $A:\mathcal{H}^p\to V$ such that $f=A\circ f_{\lambda_p}$. Clearly, A is surjective