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On the relative p-capacity

By Toshio HORIUCHI

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1. Introduction.

Let E and F denote compact and open subsets of $\boldsymbol{R}^n$, respectively, $E \subset F$. The number

$$C_p(E, F) = \inf\left\{\int_F |\nabla u|^p dx : u \in C_E(F)\right\} \tag{1.1}$$

is called the p-capacity of a compactum E relative to F. Here $p \geqq 1$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)$ and $C_E(F)$ is the class of functions $u(x) \in C^{0,1}(F)$ with $u(x) \geqq 1$ for $x \in E$ and compact support contained in F. For the detailed, see § 2.

The purpose of this paper is to study the p-capacity $C_p(E, F)$ and make clear its behavior as a set function from the point of view of the relativity of E and F. For $p>1$ we shall show that the p-capacity of E relative to F can not remain bounded when E fills up F, or equivalently, when F shrinks away to E, if and only if $C_p(E, F)>0$. In other words, the p-capacity of the whole space F is naturally considered $+\infty$ provided $p>1$. By making use of this fact, we can give simple proofs of metric properties of the p-capacity in terms of Hausdorff measure, most of which are already known but the proofs in this paper seem to be more direct than those based on the non-linear potential theory initiated by V.G. Maz'ja and V.P. Havin [**13**], [**14**], N.G. Meyers [**15**]. This theory has been extensively developed during the last decade to fill the gap to a certain extent between the classical potential theory and non-linear counterparts of Newton and Riesz capacities (See [**4**], [**6**], [**9**] and [**10**]). However our methods in this paper are not based on potential theory but on the effective use of the theory of the Dirichlet problem for non-linear elliptic differential equations and the imbedding theorems of Sobolev type. Roughly speaking, Theorem 3.1 stated in § 3 and the Sobolev imbedding theorem give the upper and lower estimates for the p-capacity respectively. It is interesting that the methods in this paper can be applied to the study of the degenerated elliptic equations as well (See [**11**]).

Here we note that H. Federer and W.P. Ziemer also presented in [**8**] a direct treatment of this topic for $F = \boldsymbol{R}^n$, which was based on geometric measure theory. For the complete references, see the book by V.G. Maz'ja [**12**] (See

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