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On hypoellipticity for a certain operator with double characteristic

Dedicated to Professor Mutsuhide Matsumura on his 60th birthday

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§1. Introduction and result.

In this paper, we consider C^∞-hypoellipticity for the operator

$$P=D_1^2+D_2^2+D_3^2+x_3^2D_4^2+(f(x_1)-1)D_4, \tag{1.1}$$

$$\left(D_j=-i\frac{\partial}{\partial x_j} \qquad j=1,2,3,4\right)$$

in neighborhoods ($\subset \boldsymbol{R}^4$) of the hypersurface $x_1=0$. Here we assume that the function $f(x_1)$ has the following properties:

(A.1) (i) $f(0)=0$, $\quad f(x_1)>0 \quad$ *if* $x_1\neq 0$.

(ii) $f(x_1)$ *is monotone in the intervals* $[0, \delta)$ *and* $(-\delta, 0]$ *for some* $\delta>0$.

Notice that the above operator (1.1) is a degenerate elliptic operator with double characteristic $\Sigma=\{(x, \xi)\in T^*\boldsymbol{R}^4\setminus 0;\ \xi_1=\xi_2=\xi_3=x_3=0\}$. Also notice that the canonical symplectic form $\sigma=\sum_j dx_j\wedge d\xi_j$ is of constant rank ($=2$) on $T_\rho\Sigma$ for any point $\rho\in\Sigma$. A. Grigis [**3**] treated a class of such operators after the important work of L. Boutet de Monvel [**1**]. He has given a condition which is necessary and sufficient for them to be hypoelliptic with loss of one derivative. Roughly speaking, his condition is that Melin's invariant ($=$subprincipal symbol $+$positive trace/2) does not take non-positive (real) values on the characteristic manifold Σ. For the operator (1.1), it becomes $0<f(x_1)<2$ if $\mathcal{I}m\ f(x_1)=0$ (*cf.* the condition (b) of théorème 0.1 in [**3**]). So, under the assumption (i) of (A.1), the operator (1.1) does not satisfy the condition on the hypersurface $x_1=0$. Nevertheless, it has a possibility to be hypoelliptic with loss of more than one derivatives.

First, let us give a condition of non-hypoellipticity for the operator (1.1):

THEOREM 1. *In addition to the hypothesis* (A.1), *we assume that*

(A.2) *there exist positive numbers* δ_1 *and* ε *such that*

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