J. Math. Soc. Japan
Vol. 43, No. 3, 1991

Some note on Gevrey hypoellipticity and solvability on torus

By Todor GRAMCHEV, Petar POPIVANOV and Masafumi YOSHINO

(Received Aug. 1, 1990)
(Revised Aug. 6, 1990)

1. Introduction.

In this paper, we are concerned with the hypoellipticity and the solvability in a Gevrey class. Let G^{σ} be a Gevrey class of order σ, $1 \leqq \sigma \leqq \infty$. For a domain Ω in $\boldsymbol{R}^n$ and a differential operator P on Ω, P is said to be (G^{σ}-) hypoelliptic in Ω if, for any subdomain $\Omega' \subset \Omega$ and any distribution $u \in D'(\Omega')$ such that $Pu \in G^{\sigma}(\Omega')$ it follows that $u \in G^{\sigma}(\Omega')$. We say that P is (G^{σ}-) globally hypoelliptic in Ω if, for any $u \in D'(\Omega)$ such that $Pu \in G^{\sigma}(\Omega)$ it follows that $u \in G^{\sigma}(\Omega)$. Though hypoelliptic operators are globally hypoelliptic, there are considerable gaps between these notions. In fact, the hyperbolic operator $(\partial/\partial x_1 - \tau\partial/\partial x_2)^2$ on $\boldsymbol{T}^2 = \boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$, $x = (x_1, x_2) \in \boldsymbol{T}^2$ is globally hypoelliptic for some real value τ (cf. Greenfield-Wallach [**4**]).

In Section 2, we discuss solvability, hypoellipticity and non hypoellipticity in G^{σ} for operators satisfying a resonance type condition ($\mathcal{R}$). In fact, we give equations which are G^{σ}-globally hypoelliptic for σ near 1, and that are not G^{σ}-globally hypoelliptic for large σ. We note that such examples are known for local hypoellipticity. But, our example does not follow from such examples because global hypoellipticity does not necessarily imply local hypoellipticity.

In Section 3, we study the resonance type condition ($\mathcal{R}$) in Section 2. By Greenfield's example, we immediately see that ($\mathcal{R}$) is not necessary for the global hypoellipticity, in general. Instead of ($\mathcal{R}$), we introduce a so-called Siegel condition in G^{σ} (cf. (3.2)), which is weaker than ($\mathcal{R}$). We consider equations treated in [**16**], namely, equations satisfying the condition (A.1) which follows. Then, the Siegel condition is necessary and sufficient for G^{σ}-global hypoellipticity. We also give an example such that we cannot drop (A.1), in general (cf. Remark 3.3). We also note that hyperbolic equations as well as elliptic equations may satisfy the condition (3.2).

The last section is devoted to the study of the solvability of semilinear equations. As applications of the arguments in §§ 3 and 4, we show the exis-