J. Math. Soc. Japan
Vol. 43, No. 3, 1991

Dedekind sums and quadratic residue symbols of imaginary quadratic fields

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(Received July 10, 1990)

§ 1. Introduction.

Let K be an imaginary quadratic field embedded in the complex number field $\boldsymbol{C}$ and $\mathcal{O}_K$ its ring of integers. Consider the subgroup $\Gamma(8)$ of $SL(2, \mathcal{O}_K)$ consisting of all matrices congruent to $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ modulo 8. As was noticed by Kubota [7], the map $\chi: \Gamma(8) \to \boldsymbol{Z}/2\boldsymbol{Z}$ defined by

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \dfrac{1}{2}\left(1-\left(\dfrac{c}{a}\right)\right), & c \neq 0 \\ 0\,, & c=0 \end{cases}$$

is a homomorphism, the homomorphic property being essentially equivalent to the reciprocity law of the quadratic residue symbol (c/a) of K. On the other hand, by a result of Sczech [10], we have homomorphisms from $\Gamma(8)$ to the additive group of $\boldsymbol{C}$ explicitly given by generalized Dedekind sums. The aim of this paper is to study the relation between these two kinds of homomorphisms. The main result is that there exists, among the linear combinations of Sczech's homomorphisms, a homomorphism Ψ with values in the ring $\boldsymbol{Z}$ of rational integers such that

$$\chi(A) \equiv \Psi(A) \pmod 2 \tag{1}$$

for every $A \in \Gamma(8)$. This was conjectured in [10].

To be more specific, let L be a lattice in $\boldsymbol{C}$ and denote by $\mathcal{O}_L$ the ring consisting of all m in $\boldsymbol{C}$ with $mL \subset L$. Let, for z in $\boldsymbol{C}$ and a non-negative integer n,

$$E_n(z) = \sum_{\substack{w \in L \\ w+z \neq 0}} (w+z)^{-n} |w+z|^{-s}|_{s=0}\,,$$

where the value at $s=0$ is to be understood in the sense of analytic continuation. Put, for two integers a, c in $\mathcal{O}_L$ with $c \neq 0$,

$$D(a, c) = \frac{1}{c} \sum_{m \in L/cL} E_1\left(\frac{am}{c}\right) E_1\left(\frac{m}{c}\right)$$