J. Math. Soc. Japan
Vol. 43, No. 3, 1991

Complete metrics of negative Ricci curvature

By O. KOBAYASHI and T. GOMIKAWA

(Received May 1, 1990)

Gao and Yau have constructed metrics of negative Ricci curvature on every compact 3-manifold ([1], [2], [3]). They however used techniques peculiar to 3-manifolds and it is hard to see how their method is applicable to general higher dimensional manifolds. In this paper we use simple triangulation argument to construct metrics of negative Ricci curvature on the complement of a point, which will be a partial evidence for affirmative answer to the question whether every manifold with dimension $\geqq 3$ can admit a metric with negative Ricci curvature (Problem 24 of [4]).

THEOREM. *For any connected closed manifold M of dimension $\geqq 2$ and a point p of M, $M \setminus \{p\}$ admits a complete metric of negative Ricci curvature.*

Note that the conclusion is false if Ricci curvature is replaced by sectional curvature. For example, take $M=\boldsymbol{R}P^n$, $n \geqq 3$.

§ 1. Preliminaries

LEMMA. *Let g and $\bar{g}$ be metrics on an n-manifold which are conformaly related as $\bar{g}=e^{-2u}g$ for some smooth function u. Then,*

$$\mathrm{Ric}(\bar{g}) \leqq (n-2)\nabla^2 u+(\Delta u)g+\mathrm{Ric}(g)\,, \tag{1}$$

where Hessian etc. in the right side are taken with respect to g. Assume further that $n \geqq 2$, $u=u(t)$ for some other function t and that $\ddot{u}=(d/dt)^2 u \leqq 0$. Then,

$$\mathrm{Ric}(\bar{g}) \leqq \ddot{u}\,|dt|^2 g+\dot{u}((n-2)\nabla^2 t+(\Delta t)g)+\mathrm{Ric}(g)\,. \tag{2}$$

PROOF. Both inequalities follow immediately from the formula; $\mathrm{Ric}(\bar{g})=(n-2)\nabla^2 u+(\Delta u)g+(n-2)(du \otimes du-|du|^2 g)+\mathrm{Ric}(g)$.

PROPOSITION 1. *Let D be a d-dimensional disk in $\boldsymbol{R}^n$ and g a metric of $\boldsymbol{R}^n$. Suppose $n>d$, $n \geqq 2$ and $\mathrm{Ric}(g)<0$ in a neighborhood of ∂D. Then, there exists another metric $\bar{g}$ such that $\bar{g}=g$ near ∂D and $\mathrm{Ric}(\bar{g})<0$ in a neighborhood of D.*

PROOF. Put $D(r)=\{(x, 0) \in \boldsymbol{R}^d \times \boldsymbol{R}^{n-d}\,;\, |x|<r\} \subset \boldsymbol{R}^n$, where $|x|=(\sum_{j=1}^d (x_j)^2)^{1/2}$. We may assume $D=D(3)$ and $\mathrm{Ric}(g)<0$ on $D \setminus D(1)$.