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Remarks on connections between the Leopoldt conjecture, p-class groups and unit groups of algebraic number fields

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Introduction.

Let p be a prime number. Leopoldt [8] showed that the p-adic rank r_p of the unit group of a totally real abelian number field K equals the number of non-trivial characters of K such that the p-adic L-functions associated to them have not value 0 at 1. Moreover, he obtained the p-adic class number formula in case where the p-adic rank equals the total number of non-trivial characters which is equal to the rank of the unit group. The Leopoldt conjecture comes from this. This equality of the p-adic rank and the rank of the unit group for an abelian field was verified by Ax [1] for several special cases, and was proved completely by Brumer [2] in the general case.

We define the p-adic rank of the unit group of an algebraic number field to which we refered above. Let $\mathcal{O}$ be an integral domain and $\mathcal{K}$ be its field of quotients. For an $\mathcal{O}$-module M, we define the essential $\mathcal{O}$-rank of M to be the value of $\dim_{\mathcal{K}} M\otimes_{\mathcal{O}}\mathcal{K}$, and denote it by ess. $\mathcal{O}$-rank M.

Let k denote a finite algebraic number field throughout this paper. Let E_1 be the group of units which are congruent to 1 modulo every prime $\mathfrak{p}$ lying over p, and let $U_{\mathfrak{p}}(1)$ be the group of the local units u such that $u\equiv 1 \bmod \mathfrak{p}$. Then E_1 is embedded into $\prod_{\mathfrak{p}|p}U_{\mathfrak{p}}(1)$ by $\varepsilon\rightarrow(\varepsilon, \varepsilon, \cdots, \varepsilon)$. Denote by $\bar{E}_1$ the closure of E_1 in $\prod U_{\mathfrak{p}}(1)$. Since $U_{\mathfrak{p}}(1)$ are multiplicative $\boldsymbol{Z}_p$-modules, where $\boldsymbol{Z}_p$ is the ring of p-adic integers, $\bar{E}_1$ is also a $\boldsymbol{Z}_p$-module. We refer to the ess. $\boldsymbol{Z}_p$-rank of $\bar{E}_1$ as the p-adic rank of the unit group of k, and denote it by r_p in this paper.

The Leopoldt conjecture predicts that the p-adic rank equals the essential $\boldsymbol{Z}$-rank of the unit group in any algebraic number field. We know by Brumer [2] that this equality holds for an abelian extension of an imaginary quadratic number field, and also know by Miyake [10] for certain non-abelian extensions of imaginary quadratic number fields.

Let r be the essential $\boldsymbol{Z}$-rank of the unit group of k, and we set $\delta_p=r-r_p$.