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On the Milnor number of a generic hyperplane section

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§ 0. Introduction.

Let $F(z_1, \cdots, z_n)$ be an analytic function on an open neighbourhood of the origin $\vec{0}$ in $\boldsymbol{C}^n$ with $F(\vec{0})=0$ and let $V=F^{-1}(0)$. Suppose that $F(z)$ has an isolated critical point at the origin. Then for sufficiently small $\varepsilon>0$, the map $\phi: S_\varepsilon - K_\varepsilon \to S^1$ which is defined by $\phi(z)=F(z)/|F(z)|$ gives a smooth fiber bundle, which is called the Milnor fibration. Here $S_\varepsilon=\{z\in \boldsymbol{C}^n \mid |z|=\varepsilon\}$, $K_\varepsilon=S_\varepsilon \cap V$ and $S^1=\{z\in \boldsymbol{C} \mid |z|=1\}$. Moreover the fiber $X_t=\phi^{-1}(t)$ is an $(n-2)$-connected $2(n-1)$-dimensional smooth manifold and has the homotopy type of a bouquet $S^{n-1}\vee \cdots \vee S^{n-1}$ of $(n-1)$-spheres ([3]). $\mu^{(n)}=$the $(n-1)$-*th* Betti number of X_t is usually called *the Milnor number of* F (or V). It is important to calculate the Milnor number in order to study topological properties of V. Suppose that F is non-degenerate and convenient, then the beautiful formula by Kouchnirenko ([2]) says that $\mu^{(n)}=\nu^{(n)}$, where $\nu^{(n)}$ is the *Newton number of* F (§ 1). By this formula, we can calculate the Milnor number via the Newton boundary of F.

Let $L=\{z_n=a_1z_1+\cdots+a_{n-1}z_{n-1}\}$ be a generic hyperplane through the origin $\vec{0}$. $V\cap L=f^{-1}(0)$ is called *a generic hyperplane section*, where $f(z_1, \cdots, z_{n-1})=F(z_1, \cdots, z_{n-1}, a_1z_1+\cdots+a_{n-1}z_{n-1})$. f has also an isolated critical point at the origin and its Milnor number $\mu^{(n-1)}$ is independent of the choice of L. Similarly $\mu^{(i)}$ $(1\leqq i\leqq n-1)$ can be defined and we define μ^* by $\mu^*=(\mu^{(n)}, \mu^{(n-1)}, \cdots, \mu^{(1)})$. It is known that μ^* is determined by F ([7]). However it is not known *how* μ^* *can be calculated for a given* F. Because, even if F is non-degenerate, f is not necessarily non-degenerate. Hence we cannot apply Kouchnirenko's formula even to $\mu^{(n-1)}$. If f is degenerate, then $\mu^{(n-1)}\geqq\nu^{(n-1)}$ ([2]) and similarly $\mu^{(i)}\geqq\nu^{(i)}$ $(1\leqq i\leqq n-1)$. Thus in order to calculate μ^*, we want to know *how the degeneracy index* $\alpha^{(i)}=\mu^{(i)}-\nu^{(i)}$ ([4]) *is determined by* F. In this paper, we will show the following result.

THEOREM A. *Let* $F(x, y, z)$ *be an analytic function on an open neighbourhood of the origin in* $\boldsymbol{C}^3$ *with* $F(\vec{0})=0$. *Suppose that* F *has an isolated critical point at the origin and that* F *is non-degenerate and convenient. Let* $z=ax+by$ *be a generic hyperplane and let* $f(x, y)=F(x, y, ax+by)$. *Let* $\mu^{(2)}$ *and* $\nu^{(2)}$ *be the Milnor number and the Newton number of* f *respectively. Then the degeneracy*