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On the geometry of projective immersions

By Katsumi NOMIZU* and Ulrich PINKALL

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In two preceeding papers [5] and [6] we have given a new general approach to classical affine differential geometry and established the basic results concerning the geometry of affine immersions. The purpose of the present paper is to begin the study of projective immersions. We shall concentrate our attention to the case of codimension one.

In Section 1 we recall the notion of projective structure on a manifold and state relevant facts. In Section 2 we define the notion of projective and equiprojective immersions and related concepts such as totally geodesic and umbilical immersions. In Section 3 we study equiprojective immersions of a flat projective structure (M, P) into a flat projective structure $(\tilde{M}, \tilde{P})$ of one higher dimension and show that they are umbilical, provided $\dim M \geqq 3$ and the rank of $h \geqq 2$. We derive certain corollaries and determine all connected, compact, umbilical hypersurfaces in $\boldsymbol{R}P^{n+1}$. In Section 4 we prove the projective version of the theorem of Berwald which characterizes quadrics in affine differential geometry. In Section 5 we study the effect of a projective change of the ambiant connection on a nondegenerate hypersurface M, namely, how the affine normal, the Blaschke induced connection, the affine metric, and the cubic form change. We find that the difference tensor between the Blaschke connection and the Levi-Civita connection is a projective invariant. We hope to find some more applications of these formulas in the study of nondegenerate hypersurfaces in $\boldsymbol{R}P^{n+1}$.

1. Projective structure.

We recall from [4] the notion of projective structure P on a differentiable manifold M. It is defined by an atlas of local affine connections $(U_\alpha, \nabla_\alpha)$, where $\{U_\alpha\}$ is an open covering of M and ∇_α is a torsion-free affine connection on U_α such that in any nonempty intersection $U_\alpha \cap U_\beta$ the connections ∇_α and ∇_β are projectively equivalent. Here, in general, two affine connections ∇ and $\bar{\nabla}$ are said to be projectively equivalent if there is a 1-form μ such that

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