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A formula for the logarithmic derivative of Selberg's zeta function

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1. Introduction.

Let M be a compact Riemann surface of genus $g \geqq 2$. The universal covering surface of M is conformally equivalent to the upper half plane H. Therefore M is the quotient of H by the discontinuous group Γ, consisting of, apart from the identity, hyperbolic transformations. We assume that the Gaussian curvature of M is normalized to be -1. It is known that the area $\mathcal{A}(M)$ of M is equal to $4\pi(g-1)$ via the Gauss-Bonnet theorem. Let T be a finite-dimensional unitary representation of Γ, and let χ be its character. Let Δ denote the Laplacian for H. Let $\{\lambda_n(\chi)\}_{n=0,1,2,\cdots}$ be the sequence of distinct eigenvalues corresponding to the problem $\Delta F + \lambda F = 0$ on M, where the eigenfunction $F(x)$ is required to transform under Γ by $F(\gamma x) = T(\gamma)F(x)$. We denote by $m_n(\chi)$ the multiplicity of $\lambda_n(\chi)$. It is well known that the eigenvalues are all real and non-negative, and that the set of such eigenfunctions is complete in the space consisting of those measurable functions on H which transform in this manner, and square integrable over a fundamental domain of Γ. Associate with the sequence $0 \leqq \lambda_0(\chi) < \lambda_1(\chi) < \cdots$ of eigenvalues, a sequence, consisting of those numbers $r_n(\chi)$ that satisfy the equation $\lambda_n(\chi) = 1/4 + r_n(\chi)^2$ $(n=0, 1, 2, \cdots)$. From this it follows that $r_n(\chi)$ is either real or pure imaginary. We choose and fix $r_n(\chi)$ so that when it is real, we have $r_n(\chi) \geqq 0$, and when it is pure imaginary, we have $\sqrt{-1}r_n(\chi) < 0$. In the convention of notation, we put $\lambda_0(\chi) = 0$ and $r_0(\chi) = \sqrt{-1}/2$, and we denote by $m_0(\chi) \geqq 0$ the multiplicity of the possible eigenvalue $\lambda_0(\chi) = 0$ throughout this paper.

By assumption on Γ, each $\gamma \in \Gamma$ $(\gamma \neq e)$ is conjugate in $PSL(2, \boldsymbol{R})$ to a unique transformation of the form $z \mapsto e^{u_\gamma} z$, where u_γ is a positive real number. Clearly u_γ depends only on the conjugacy class. We will denote by $\{\gamma\}$ the conjugacy class corresponding to γ within Γ itself and by $\{\Gamma\}$ the set of all Γ-conjugacy classes in Γ. It is known that the numbers $\{u_\gamma; \{\gamma\} \in \{\Gamma\} \setminus \{e\}\}$ are bounded away from zero. We choose and fix ε_0 so small that it is smaller than these numbers throughout the paper. An element $\gamma \in \Gamma$ $(\gamma \neq e)$ is called primitive if