J. Math. Soc. Japan
Vol. 40, No. 1, 1988

A theorem on the outradii of Teichmüller spaces

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

By Toshihiro NAKANISHI

(Received June 2, 1986)

1. Introduction.

The purpose of this paper is to present some results related to the Teichmüller spaces. Let Γ be a Fuchsian group acting on the upper half plane $\boldsymbol{U}=\{\mathrm{Im}\, z>0\}$. Then the Teichmüller space $\boldsymbol{T}(\Gamma)$ is represented as a bounded domain in the Banach space $\boldsymbol{B}(\boldsymbol{U}^*, \Gamma)$ of bounded quadratic differentials for Γ in the lower half plane $\boldsymbol{U}^*$ (Bers [1]). We consider the function $\varphi_\alpha(z)=\alpha z^{-2}$, $\alpha\in\boldsymbol{C}$, defined in $\boldsymbol{U}^*$. Let F_α be a solution of the differential equation $\{f, z\}=\varphi_\alpha(z)$, where $\{f, z\}=(f''/f')'-(1/2)(f''/f')^2$ denotes the Schwarzian derivative of f. Then it is known that F_α is univalent in $\boldsymbol{U}^*$ if and only if α belongs to the set $\boldsymbol{V}=\{\alpha=(1-re^{2i\theta})/2\,;\ r\leqq 4\cos^2\theta,\ 0\leqq\theta<\pi\}$ ([4, 5]). Since it has such a simple form, the function φ_α, $\alpha\in\boldsymbol{V}$, cannot belong to $\boldsymbol{T}(\Gamma)$ unless Γ is one of some elementary groups (see Section 4). However if we are allowed to vary Γ in its quasiconformal equivalence class, we obtain the following result:

THEOREM A. *Let $Q_U(\Gamma)$ be the set of all quasiconformal automorphisms of U compatible with Γ. If Γ contains a hyperbolic element, then for each $\alpha\in\boldsymbol{V}$ there exists a sequence w_n, $n=1, 2, \cdots$, in $Q_U(\Gamma)$ with an element $\varphi_n\in\boldsymbol{T}(w_n\circ\Gamma\circ w_n^{-1})$ such that φ_n converges normally (uniformly on every compact subsets of $\boldsymbol{U}^*$) to φ_α in $\boldsymbol{U}^*$.*

The motivation of this theorem originates from a problem related to the outradii of Teichmüller spaces. By a theorem of Nehari [8] the outradius $\boldsymbol{o}(\Gamma)$ of $\boldsymbol{T}(\Gamma)$ does not exceed 6. The following theorem shows that this value 6 is sharp within the range of the quasiconformal equivalence class.

THEOREM B. *Set $\mathcal{O}(\Gamma)=\sup\{\boldsymbol{o}(w\circ\Gamma\circ w^{-1})\,;\ w\in Q_U(\Gamma)\}$. Then the equality $\mathcal{O}(\Gamma)=6$ holds if $0<\dim\boldsymbol{T}(\Gamma)$.*

Actually if Γ is of the second kind, Theorem B is trivially deduced from

This research was partially supported by Grant-in-Aid for Scientific Research (No. 60740108), Ministry of Education, Science and Culture.