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Existence of a non-inductive linear form on certain solvable Lie algebras

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Introduction.

Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{g}^*$ its dual vector space. Given $f \in \mathfrak{g}^*$, we set

$$\mathfrak{g}(f) = \{x \in \mathfrak{g} ;\ f([x, y]) = 0 \quad \text{for all } y \in \mathfrak{g}\}.$$

Then, $\mathfrak{g}(f)$ acts naturally on $\mathfrak{g}/\mathfrak{g}(f)$. We say that f is *inductive* if for every $x \in \mathfrak{g}(f)$, the operator $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}(f)} x$ is nilpotent. So, if f is inductive, the linear Lie algebra $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}(f)} \mathfrak{g}(f)$ is nilpotent by Engel's theorem. We refer the reader to the papers Poguntke [**6**, Lemma 2] and Tauvel [**10**, Lemme 3.1] for various equivalent conditions for the inductivity of linear forms (we note that although the base field k is assumed to be algebraically closed throughout [**10**], the proof of Lemme 3.1 in that paper still works for $k = \boldsymbol{R}$).

Now let $\mathfrak{g}$ be exponential and $G = \exp \mathfrak{g}$ the corresponding connected and simply connected Lie group. Denote by $\hat{G}$ the equivalence classes of irreducible unitary representations of G with the Fell topology. We equip the finite dimensional vector space $\mathfrak{g}^*$ with the natural topology and the coadjoint orbit space $\mathfrak{g}^*/G$ with the quotient topology. Then, one knows that the Kirillov-Bernat mapping $\rho : \mathfrak{g}^*/G \to \hat{G}$ is a continuous bijection and it is a long-standing conjecture that ρ is a homeomorphism. Among several works toward this conjecture (cf. Fujiwara [**2**] and its Introduction), Boidol [**1**] made inductive linear forms play a significant role as follows.

THEOREM (Boidol). *Let $G = \exp \mathfrak{g}$ be an exponential Lie group. If every linear form on $\mathfrak{g}$ is inductive, ρ is a homeomorphism.*

Thus there arises a natural question: to what extent does the above Boidol's theorem cover the exponential Lie groups? This motivated the present work and the purpose of this note is to provide a class of completely solvable (hence exponential) Lie algebras $\mathfrak{s}$ on which there is always a non-inductive linear form. So, the Boidol's theorem is not applicable for the solvable Lie groups $S = \exp \mathfrak{s}$. Furthermore, by a theorem of Poguntke [**6**, Theorem 10] (for our S, Theorem 3 in [**5**] suffices), the involutory Banach algebra $L^1(S)$ is not sym-