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Some new algebraic cycles on Fermat varieties

By Noboru AOKI

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Introduction.

Let X_m^n be the Fermat variety of dimension n (n=even) and degree m defined over $\boldsymbol{C}$, that is, a hypersurface in the projective space $\boldsymbol{P}_{\boldsymbol{C}}^{n+1}$ defined by the diagonal equation:

$$x_0^m+x_1^m+\cdots+x_{n+1}^m=0.$$

Let μ_m be the group of m-th root of unity and set $G_m^n=(\mu_m)^{n+2}/\text{diagonal}$. Then G_m^n acts on X_m^n and its character group $\hat{G}_m^n$ can be identified with the following group:

$$\left\{(a_0, a_1, \cdots, a_{n+1})\in(\boldsymbol{Z}/m)^{n+2} \;\middle|\; \sum_{i=0}^{n+1} a_i=0\right\}$$

by setting $\alpha(g)=\zeta_0^{a_0}\cdots\zeta_{n+1}^{a_{n+1}}$ for any $\alpha=(a_0, \cdots, a_{n+1})\in\hat{G}_m^n$ and any $g=(\zeta_0, \cdots, \zeta_{n+1})\in G_m^n$.

As for the cohomology group of X_m^n, the following results are well known (see [3], [4]):

$$H_{\text{prim}}^n(X_m^n, \boldsymbol{C})=\bigoplus_{\alpha\in\mathfrak{A}_m^n} V(\alpha), \qquad \dim V(\alpha)=1,$$

$$(H^{r,r}(X_m^n)\cap H_{\text{prim}}^n(X_m^n, \boldsymbol{Q}))\otimes\boldsymbol{C}=\bigoplus_{\alpha\in\mathfrak{B}_m^n} V(\alpha),$$

where $V(\alpha)=\{\xi\in H^n(X_m^n, \boldsymbol{C}) \mid g^*(\xi)=\alpha(g)\xi, \forall g\in G_m^n\}$ and $r=n/2$. The index sets $\mathfrak{A}_m^n$ and $\mathfrak{B}_m^n$ are defined as follows:

$$\mathfrak{A}_m^n=\{(a_0, \cdots, a_{n+1})\in\hat{G}_m^n \mid a_i\neq 0 \text{ for every } 0\leqq i\leqq n+1\},$$

$$\mathfrak{B}_m^n=\left\{(a_0, \cdots, a_{n+1})\in\mathfrak{A}_m^n \;\middle|\; \sum_{i=0}^{n+1}\langle ta_i/m\rangle=n/2+1, \forall t\in(\boldsymbol{Z}/m)^\times\right\},$$

where, for $a\in\boldsymbol{Z}/m$, $\langle a/m\rangle$ expresses the unique rational number such that $0\leqq\langle a/m\rangle<1$, $m\cdot\langle a/m\rangle\equiv a \pmod m$.

The Hodge conjecture for X_m^n asserts that the following claim is true for every $\alpha\in\mathfrak{B}_m^n$.

CLAIM(α): *$V(\alpha)$ is generated by the cohomology classes of algebraic cycles on X_m^n.*