J. Math. Soc. Japan
Vol. 39, No. 2, 1987

Gauss-Manin connection of integral of difference products

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

By Kazuhiko AOMOTO

(Received Aug. 14, 1985)

0. Let $x_1, \cdots, x_p$ be real distinct numbers. As a function of $x_1, \cdots, x_p$ the integral

$$F(x_1, \cdots, x_p) = \int \prod_{1 \leq i < j \leq N} (x_i - x_j)^{\lambda_{i,j}} dx_{p+1} \wedge \cdots \wedge dx_N \tag{0.1}$$

for $2 \leqq p \leqq N$, over a suitable cycle satisfies an integrable analytic differential system (called Gauss-Manin connection in analytic geometry or holonomic system from micro-local point of view). *In this note we want to give an explicit formula of it.* In the sequel we denote by Φ the product $\prod_{1 \leq i < j \leq N} (x_i - x_j)^{\lambda_{i,j}}$.

Roughly speaking, our method is as follows. The structure of the integral (0.1) is of fibre type. This enables us to give a recurrent relation for integration over each variable $x_{p+1}, \cdots, x_N$ in the reverse order. Namely we first integrate (0.1) over x_N. Then we get the function of $x_1, \cdots, x_{N-1}$ satisfying a certain Gauss-Manin connection of classical Jordan-Pochhammer type. Next we do it over x_{N-1} and get a differential equation of similar nature and so on. Finally $F(x_1, \cdots, x_p)$ satisfies a Gauss-Manin connection which can be computed in inductive way.

We assume from now on that $x_1 < x_2 < \cdots < x_p$ and that the point $(x_{p+1}, \cdots, x_N)$ lies in $\boldsymbol{R}^{N-p}$. We denote by Δ the closure of any of relatively compact components of the open set: $x_{p+\nu} \neq x_j$, $1 \leqq j \leqq p$ and $x_{p+\mu} \neq x_{p+\nu}$ for $\mu \neq \nu$ in $\boldsymbol{R}^{N-p}$. If $\lambda_{i,j}$ are all positive, the integral over each domain Δ has a definite meaning. If some of $\lambda_{i,j}$ are negative we have to replace Δ by its regularized cycle Δ^{reg} (which is called "renormalized" by physicists and which is essentially the same as "finite part of divergent integrals" in the sense of J. Hadamard), such that $\int_{\Delta^{\text{reg}}}$ is an analytic continuation of the original $\int_{\Delta}$ considered as function of the variables $\lambda = (\lambda_{i,j})_{i<j}$ (For the way of construction, see [**A2**] or [**T**] pp. 314-318). The regularized cycle Δ^{reg} defines a twisted homological $(N-p)$-cycle in the affine algebraic variety $X = \boldsymbol{C}^{N-p} - \bigcup (x_i = x_j)$ where $1 \leqq i \leqq N$,