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Remarks on real nilpotent orbits of a symmetric pair

By Jiro SEKIGUCHI

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Introduction.

In this paper, we shall study the real nilpotent orbits of the vector space associated to a semisimple symmetric pair.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be its involution. Then we obtain the direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ for σ. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a symmetric pair. The first purpose of this paper is to prove a theorem concerning the H-orbital structure of the nilpotent subvariety $\mathcal{N}(\mathfrak{q})$ of $\mathfrak{q}$. The second purpose is to determine the orbital structure of $\mathcal{N}(\mathfrak{q})$ when the pair $(\mathfrak{g}, \mathfrak{h})$ is of split rank one in the sense of [**OS**, Def. 2.5.1].

We are going to explain the contents of this paper in detail. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$ acting on $\mathfrak{q}$. Then H leaves $\mathcal{N}(\mathfrak{q})$ invariant. Let $[\mathcal{N}(\mathfrak{q})]$ be the totality of H-orbits of $\mathcal{N}(\mathfrak{q})$. One can define symmetric pairs $(\mathfrak{g}^a, \mathfrak{h}^a)$ and $(\mathfrak{g}^d, \mathfrak{h}^d)$ from $(\mathfrak{g}, \mathfrak{h})$ as we did in [**OS**, §1]. Using the notation there, we define $\mathcal{N}(\mathfrak{q}^a)$ and $\mathcal{N}(\mathfrak{q}^d)$. In §1, we shall show the following theorem.

THEOREM. $[\mathcal{N}(\mathfrak{q})] \cong [\mathcal{N}(\mathfrak{q}^a)] \cong [\mathcal{N}(\mathfrak{q}^d)]$.

Now suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are the complexifications of a real semisimple Lie algebra $\mathfrak{g}_0$ and its maximal compact subalgebra $\mathfrak{k}_0$, respectively. Then $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. In this case, one finds that $(\mathfrak{g}^d, \mathfrak{h}^d)\cong(\mathfrak{g}_0\oplus\mathfrak{g}_0, \mathfrak{g}_0)$ and therefore that $\mathcal{N}(\mathfrak{q}^d)$ is identified with the totality $[\mathcal{N}(\mathfrak{g}_0)]$ of the real nilpotent orbits of $\mathfrak{g}_0$. Then the theorem mentioned above implies that $[\mathcal{N}((\mathfrak{p}_0)_C)]\cong[\mathcal{N}(\mathfrak{g}_0)]$. Here $(\mathfrak{p}_0)_C$ is the complexification of $\mathfrak{p}_0$ which is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to its Killing form. This is a modified version of an unpublished result of Kostant (see Remark 1.10 (ii)). D. Vogan pointed out the importance of this bijection in the study of irreducible representations of $\mathfrak{g}_0$. Note that in this case, $(\mathfrak{g}^a, \mathfrak{h}^a)\cong(\mathfrak{g}, \mathfrak{g}_0)$ and therefore $\mathcal{N}(\mathfrak{q}^a)\cong\mathcal{N}(\mathfrak{g}_0)$.

In [**OS**, Def. 2.5.1], the rank and the split rank of a symmetric pair were introduced. These notions correspond to the rank and split rank of a semisimple Lie algebra. In §2, we shall determine $[\mathcal{N}(\mathfrak{q})]$ in the case where $(\mathfrak{g}, \mathfrak{h})$ is irreducible and of split rank one. In this case, the structure of $[\mathcal{N}(\mathfrak{q})]$ is