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The isometry groups of manifolds admitting nonconstant convex functions

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A function $\phi: M\rightarrow \boldsymbol{R}$ on a Riemannian manifold M is by definition *convex* if, for every geodesic segment $\gamma:[a, b]\rightarrow M$, the function $\phi\circ\gamma:[a, b]\rightarrow \boldsymbol{R}$ is convex in the usual sense, i. e. $(\phi\circ\gamma)(\lambda t+(1-\lambda)s)\leqq\lambda[(\phi\circ\gamma)(t)]+(1-\lambda)[(\phi\circ\gamma)(s)]$ for all $t, s\in[a, b]$ and $\lambda\in[0,1]$. Convex functions on Riemannian manifolds arise naturally in a number of geometric contexts, and the existence of convex functions of certain types can often be used to produce information about the structure of the manifold itself. Specifically, if M is a complete Riemannian manifold and if $\phi: M\rightarrow \boldsymbol{R}$ is a convex function, then there is a C^{∞} manifold N such that $M-\{x\in M\mid\phi(x)=\inf_{M}\phi\}$ is diffeomorphic to the product manifold $N\times \boldsymbol{R}$ ([4], [5]). In particular, if the minimum set $\{x\in M\mid\phi(x)=\inf_{M}\phi\}$ is empty, then M itself is diffeomorphic to such a product $N\times \boldsymbol{R}$. It is this case of empty minimum set and with, moreover, the manifold N compact that will be considered now and throughout this paper. The C^{∞} product structure $N\times \boldsymbol{R}$ on such a manifold M is obtained as a smoothing of a topological product structure that corresponds to the level sets of ϕ; specifically, there is a homeomorphism $H: M\rightarrow N\times \boldsymbol{R}$ such that ϕ is constant on $H^{-1}(N\times\{\alpha\})$ for each $\alpha\in \boldsymbol{R}$ and $H^{-1}(N\times\{\alpha\})=\{x\in M\mid\phi(x)=$ the value of ϕ on $H^{-1}(N\times\{\alpha\})\}$.

It is not necessarily the case that such an M has a product metric structure; for instance, the function e^{x} is convex with empty minimum set on the surface of revolution $\{(x, y, z)\in \boldsymbol{R}^{3}\mid x^{2}+y^{2}=e^{x}\}$ which is not metrically a product. This surface of revolution has as cross sections the level sets of the convex function, and these increase in size as the function increases. The standard (mean curvature) formula for first variation of hypersurface area combined with the observation that the mean curvature of a convex hypersurface is nonnegative shows that this size increase phenomenon extends to the general situation, at least in the case of smooth ϕ; more precisely, if ϕ is C^{∞}, then the $(n-1)$-volume, $n=\dim M$, of the smooth submanifold $\{x\in M\mid\phi(x)=\alpha\}$ is a nondecreasing function of $\alpha, \alpha\in\phi(M)$. This result holds even when ϕ is not smooth, if $(n-1)$-volume is interpreted as $(n-1)$-Hausdorff measure ([2]). The volumes of the level sets can be far from constant, as shown in the example given. But