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Additivity of Jordan $*$-maps between operator algebras

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The addition and Jordan product in operator algebras seem to be closely related. Our aim in this paper is to present a positive answer to the following problem.

Let M be a unital C^*-algebra and N be an associative $*$-algebra. A map ϕ is said to be a Jordan $*$-map from M to N, if ϕ satisfies the following conditions (i)$\sim$(iii) [2].

(i) $\phi(x\circ y)=\phi(x)\circ\phi(y)$ for all x and y in M, where $x\circ y=(1/2)(xy+yx)$.

(ii) $\phi(x^*)=\phi(x)^*$ for all $x\in M$.

(iii) ϕ is bijective.

Can we conclude that ϕ is additive?

Unfortunately, the answer to this problem is negative in the one dimensional case, even if ϕ is uniformly continuous, as the following example shows. Let $\phi(\alpha)=\alpha|\alpha|$ for $\alpha\in \boldsymbol{C}$ (the complex number field). Then ϕ is a uniformly continuous Jordan $*$-map from $\boldsymbol{C}$ to $\boldsymbol{C}$ and it is not additive. If, however, M has a system of $n\times n$ matrix units for some $n\geqq 2$, we obtain the following:

THEOREM. *Let M be a C^*-algebra, N be an associative $*$-algebra and ϕ be a Jordan $*$-map from M to N. Suppose that M has a system of $n\times n$ matrix units for some $n\geqq 2$. Then ϕ is additive.*

In [2], additivity of a Jordan $*$-map on an AW^*-algebra with no abelian direct summand was established under the hypothesis of continuity. S. Sakai conjectured that the hypothesis of continuity is redundant (see [2]). This follows from our theorem:

COROLLARY. *Let M be a von Neumann algebra (or more generally an AW^*-algebra) which has no abelian direct summand, let N be a C^*-algebra and let ϕ be a Jordan $*$-map from M to N. Then ϕ is additive. Moreover, there exist central projections e_1, e_2, e_3, e_4 in M such that ϕ is a linear $*$-ring isomorphism on Me_1, ϕ is a linear $*$-ring antiisomorphism on Me_2, ϕ is a conjugate linear $*$-ring isomorphism on Me_3 and ϕ is a conjugate linear $*$-ring antiisomorphism on Me_4.*

Throughout this paper, we always assume that M is a unital C^*-algebra, N