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Limits on $P(\omega)$/finite

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§1. Introduction.

Define the quasi-order $\leqq^*$ on $P(\omega)$ by $x\leqq^* y$, if $x\setminus y$ is finite. $x<^* y$ means that $x\leqq^* y$ and not $y\leqq^* x$. $x\sim y$ means that $x\leqq^* y$ and $y\leqq^* x$. $x\nsim y$ means that not $x\sim y$. For any cardinal κ, a κ-sequence $X=\langle a_\alpha \mid \alpha<\kappa\rangle$ is said to be a κ-limit, if X is a $<^*$-descending sequence and, whenever $y\subset\omega$ and $\forall\alpha<\kappa$ $(y<^* a_\alpha)$, $y\sim\emptyset$. We abbreviate the statement "There is a κ-limit" by $\exists\kappa$-limit. Since $\exists\kappa$-limit holds for some cardinal κ, under the continuum hypothesis (CH), ω_1 is the unique cardinal κ such that $\exists\kappa$-limit. And, if $2^\omega=\omega_2$ holds, then the following (A), (B) and (C) are the only possible cases. (A) $\exists\omega_1$-limit$+\neg\exists\omega_2$-limit. (B) $\neg\exists\omega_1$-limit$+\exists\omega_2$-limit. (C) $\exists\omega_1$-limit$+\exists\omega_2$-limit. In fact, each of them is known to be compatible with $2^\omega=\omega_2$. If we start with a ground model of CH and add ω_2 Cohen reals, then we get a model of (A) (see [3]). The Martin's Axiom(MA)$+2^\omega=\omega_2$ implies (B). And, if we start with a ground model of (B) and add ω_1 Cohen reals, then we get a model of (C). The existence of κ-limits provides still a few problems when 2^ω is much more large. In this paper, we would like to make a contribution to this subject. Since $\exists\kappa$-limit implies $\exists\mathrm{cf}\kappa$-limit, we may restrict our interest to regular cardinals. Our result is the following.

THEOREM 1 (GCH). *Let n be a natural number. Let $\kappa_0, \cdots, \kappa_n$ and λ be regular cardinals such that $\omega_1\leqq\kappa_0<\cdots<\kappa_n\leqq\lambda$. Then, there exists a poset P which satisfies the following* (i)$\sim$(iv).

(i) *P satisfies the countable chain condition* (*the c.c.c.*).
(ii) $\Vdash_P$ "$2^\omega=\check{\lambda}$".
(iii) $\forall m\leqq n$ ($\Vdash_P$ "$\exists\check{\kappa}_m$-limit").
(iv) $\forall\theta$: *regular* ($\forall m\leqq n\,(\theta\neq\kappa_m) \Rightarrow \Vdash_P$ "$\neg\exists\check{\theta}$-limit").

The rest of the paper consists of three sections. Section 2 is for preliminaries. Sections 3 and 4 are entirely devoted to the proof of the theorem.