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Williamson Hadamard matrices and Gauss sums

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§1. Williamson Hadamard matrices.

1. Let $\mathfrak{A}$ be a rational division algebra with an antiautomorphism $\tau: \xi \rightarrow \bar{\xi}$ of period two, such that the norm $\xi\bar{\xi}$ is a positive definite quadratic form in the coefficients of ξ with respect to a basis of $\mathfrak{A}$ over $\boldsymbol{Q}$. Let $\mathfrak{O}$ be a maximal order in $\mathfrak{A}$ invariant under τ. An element ε of $\mathfrak{O}$ is called a unit if its norm $\varepsilon\bar{\varepsilon}$ equals 1. The set U of all units is finite, and is a subgroup of the multiplicative group $\mathfrak{A}^*$ of $\mathfrak{A}$.

A square matrix H of order n with entries in U is called an *Hadamard matrix* in $\mathfrak{A}$ if

$$HH^* = nI, \qquad H^* = {}^t\bar{H},$$

for the unit matrix I.

If $\mathfrak{A} = \boldsymbol{Q}$ the rational number field then $U = \{1, -1\}$ and H is a usual Hadamard matrix. If $\mathfrak{A} = \boldsymbol{Q}(i)$ the Gaussian imaginary quadratic field, then $U = \{\pm 1, \pm i\}$ and H is called a *complex Hadamard matrix.* The character table of an abelian group G of order n provides an Hadamard matrix in the cyclotomic field $\boldsymbol{Q}(\zeta_m)$, $\zeta_m = e^{2\pi i/m}$, for the exponent m of G.

In the present paper we deal with rational quaternion field, although some part of the theory is carried over to a generalized quaternion field where the center is the maximal real subfield of a cyclotomic field of order 2^s. Thus let $\mathfrak{A} = \boldsymbol{Q} + \boldsymbol{Q}i + \boldsymbol{Q}j + \boldsymbol{Q}k$ with the quaternion units $1, i, j, k$ such that

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We take the Hurwitz quaternion ring as $\mathfrak{O}$. The ring $\mathfrak{O}$ consists of quaternions $\xi = a + bi + cj + dk$ with

$$a, b, c, d \in \frac{1}{2}\boldsymbol{Z}, \qquad a \equiv b \equiv c \equiv d \pmod{1}.$$

The antiautomorphism τ assigns the quaternion conjugate $\bar{\xi} = a - bi - cj - dk$ to ξ, and $\xi\bar{\xi} = a^2 + b^2 + c^2 + d^2$. The unit group U consists of 24 elements and contains the quaternion group $U_0 = \{\pm 1, \pm i, \pm j, \pm k\}$ as a normal subgroup. It also