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Strong topological transitivity and C^*-dynamical systems

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0. Introduction.

Let T denote the action of a group H as homeomorphisms of a topological space X; then (X, H, T) is said to be topologically transitive if for each pair of non-empty open sets A, $B\subseteq X$ there exists an $h\in H$ such that $A\cap T_h(B)\neq\emptyset$ (see, for example, [**13**] Chapter 5). Following [**10**], we define the C^*-dynamical system $(\mathcal{A}, H, \tau)$ to be topologically transitive if for each pair of non-zero elements x, $y\in\mathcal{A}$ there exists an $h\in H$ such that $x\tau_h(y)\neq 0$.

The algebraic definition is particularly natural if $\mathcal{A}$ is abelian. In this case τ determines an action τ' of H as homeomorphisms of the spectrum X of $\mathcal{A}$ such that $(\tau_h x)(\omega)=x(\tau'_{h^{-1}}\omega)$, for $x\in\mathcal{A}$ and $\omega\in X$, and $(\mathcal{A}, H, \tau)$ is topologically transitive if and only if (X, H, τ') is topologically transitive.

In [**10**] the definition is given in a slightly different form. These authors require that the product $\mathcal{A}_1\mathcal{A}_2$ of each pair of non-zero τ-invariant hereditary C^*-subalgebras $\mathcal{A}_1$, $\mathcal{A}_2\subseteq\mathcal{A}$ is non-zero. This obviously follows from our definition but conversely if there exist non-zero x, $y\in\mathcal{A}$ such that $x\tau_h(y)=0$ for all $h\in H$ then the product $\mathcal{A}_1\mathcal{A}_2$ of the τ-invariant hereditary C^*-subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ generated by $\{\tau_h(x)^*\mathcal{A}\tau_h(x);\ h\in H\}$ and $\{\tau_h(y)\mathcal{A}\tau_h(y)^*;\ h\in H\}$ must be zero.

Although the foregoing definition of transitivity is quite natural there is a seemingly stronger notion which appears to be more useful. The C^*-dynamical system $(\mathcal{A}, H, \tau)$ is defined to be strongly topologically transitive if for each finite sequence $\{(x_i, y_i);\ i=1, 2, \cdots, n\}$ of pairs of elements x_i, $y_i\in\mathcal{A}$ for which

$$\sum_{i=1}^{n} x_i\otimes y_i\neq 0,$$

in the algebraic tensor product $\mathcal{A}\otimes\mathcal{A}$, there exists an $h\in H$ such that

$$\sum_{i=1}^{n} x_i\tau_h(y_i)\neq 0$$

in $\mathcal{A}$.

Clearly strong topological transitivity implies topological transitivity; it suffices to apply the strong condition to a single pair (x, y). In Section 1 we show that the two properties are equivalent if $\mathcal{A}$ is abelian or finite-dimensional. We also