J. Math. Soc. Japan
Vol. 37, No. 1, 1985

Multiply connected minimal surfaces and the geometric annulus theorem

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(Received Aug. 29, 1983)

§ 0. Introduction.

The annulus theorem, which is significant in 3-manifold topology, was first announced by Waldhausen [**15**] and proved by Cannon-Feustel [**1**]. It is stated as follows:

Let M be a compact orientable P.L. manifold with boundary and let A be a P.L. annulus. Suppose $f:(A, \partial A)\to(M, \partial M)$ is an essential P.L. map. Then there exists an essential P.L. embedding $f^*:(A, \partial A)\to(M, \partial M)$.

In this paper we prove the above annulus theorem in the smooth category, realizing it by area-minimizing minimal surfaces on a Riemannian manifold.

To this end we take an arbitrary compact Riemannian manifold M of dimension m with convex boundary ($m\geqq 3$) and solve in § 3 the energy minimizing problem for essential (i.e. incompressible and boundary incompressible) maps from $(\Delta, \partial\Delta)$ into $(M, \partial M)$, where Δ is a k-ply connected compact planar domain ($2\leqq k<\infty$). Our variational problem is, as found from this setting, what is called a free boundary problem. Since the convergence in free boundary cases is attended with some troubles, we solve an appropriate fixed boundary problem in § 2 to make sure of some converging sequence whose limit is the required solution.

Next in § 4 we suppose $m=3$ and show that the above minimally immersed solution surface is an embedding or a double covering map of an embedded Möbius strip in case $k=2$. We find difficulties in fulfilling this, since the tower construction does not preserve the boundary incompressibility. We get over this obstacle by utilizing, with the characters of an annulus, a certain covering which is adequate for this situation. A simple example (Example 1) shows that the solution surfaces, in case $k\geqq 3$, are neither embeddings nor double covering maps of embedded surfaces in general. Our main result or the geometric annulus theorem is achieved in § 5:

THEOREM (Geometric Annulus Theorem). *Let M be a compact orientable Riemannian 3-manifold with convex incompressible boundary and let A be a smooth annulus. Suppose that there is an essential smooth map $f:(A, \partial A)\to(M, \partial M)$. Then:*