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Involutive automorphisms of root systems

By André HECK

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§1. Introduction and notation.

In this paper we study involutive automorphisms of reduced root systems using the following notations and definitions (patterned after those in [3], [5]).

Let Δ be a reduced root system spanning a finite-dimensional Euclidean space E with Weyl group invariant inner product $(\cdot|\cdot)$. Let Π be a fundamental system of Δ. We endow the space E with a partial ordering $\geqq$ with respect to Π: for α, β in E $\alpha\geqq\beta$ if $\alpha-\beta$ is a linear combination of roots in Π with integral non-negative coefficients. Since the Weyl group W of Δ acts simply transitively on the set of fundamental systems of Δ, there exists a unique element w_{Π} in W such that $w_{\Pi}(\Pi)=-\Pi$. The automorphism op_{Π} defined by $\mathrm{op}_{\Pi}:=-w_{\Pi}$ is called the *opposition involution* of Δ with respect to Π.

Now let σ be an involutive automorphism of Δ; denote its linearization to a transformation of E by σ too. We renorm the space E in such a way that σ extends to a congruence of E. We can decompose E into a direct sum of subspaces $E_0:=\{\alpha\in E\,|\,\sigma\alpha=-\alpha\}$ and $\bar{E}:=\{\alpha\in E\,|\,\sigma\alpha=\alpha\}$. Let $\bar{\ }: E\ni\alpha\mapsto\bar{\alpha}\in\bar{E}$ be the canonical projection of E onto $\bar{E}$ with respect to E_0. We define $\Delta_0:=\Delta\cap E_0$, $\Pi_0:=\Pi\cap\Delta_0$, $\bar{\Delta}:=\{\bar{\alpha}\,|\,\alpha\in\Delta\setminus\Delta_0\}$ and $\bar{\Pi}:=\{\bar{\rho}\,|\,\rho\in\Pi\setminus\Pi_0\}$; $\bar{\Delta}$ is called the *system of restricted roots.* The set $\tilde{\Delta}:=\{\psi\in\bar{\Delta}\mid\psi$ is not of the form $c\eta$ with $\eta\in\bar{\Delta}$, $c\in\boldsymbol{R}$, $c>1\}$ is the *system of reduced restricted roots.* In general neither $\bar{\Delta}$ nor $\tilde{\Delta}$ is a root system.

We call Π *σ-fundamental* if $\sigma\rho>0$ for each root ρ in $\Pi\setminus\Pi_0$. Throughout this paper we will assume Π to be a σ-fundamental system of Δ and call the corresponding partial ordering of E a *σ-ordering.* In §2 we state some basic properties of σ-fundamental systems and we will also give a diagrammatic description of the action of the involutive automorphism σ on Δ by introduction of a so-called *Satake diagram* of σ with respect to a σ-fundamental system Π.

We define $W_{\sigma}:=\{w\in W\,|\,w\circ\sigma=\sigma\circ w\}$, $\bar{w}:=$the restriction of an element w in W_{σ} to $\bar{E}$ and $\bar{W}:=\{\bar{w}\,|\,w\in W_{\sigma}\}$. Schattschneider [6] studied the action of a general automorphism group G on the root system of a semisimple algebraic group and determined under which conditions $\tilde{\Delta}$ is a root system with Weyl group $\bar{W}$. In §3 we will give easier proofs of these results in our less general context of $G=\{1, \sigma\}$. In §4 we show that the property of $\tilde{\Delta}$ being a root