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Capacities and Bergman kernels for Riemann surfaces and Fuchsian groups

Dedicated to Professor Yûsaku Komatu on his 70th birthday

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1. Riemann surfaces.

Let Ω be a Riemann surface with $\Omega \notin O_G$. Let $k_0(w, \omega)dw d\bar{\omega}$ denote the Bergman kernel of the Hilbert space of square integrable abelian differentials $a(w)dw$ on Ω. It has the reproducing property

$$a(\omega)=\frac{1}{\pi}\iint_{\Omega} a(w)\overline{k_0(w, \omega)}du\,dv\,. \tag{1.1}$$

We use the notation of Ahlfors and Sario [**1**, p. 302] which differs from that of Sario and Oikawa [**7**, p. 104] by a factor π.

Let $c_\beta(\omega)|d\omega|$ denote the capacity metric of the ideal boundary of Ω [**7**, p. 55]. If $g(w, \omega)$ denotes the Green's function of Ω with pole at ω then

$$g(w, \omega)=-\log|w-\omega|-\log c_\beta(\omega)+o(1) \qquad \text{as} \quad w\to\omega\,. \tag{1.2}$$

The second author [8] conjectured that

$$k_0(\omega, \omega)\geqq c_\beta(\omega)^2 \qquad \text{for} \quad \omega\in\Omega \tag{1.3}$$

and proved this for the special case that Ω is a doubly connected plane domain.

We shall prove a weaker inequality. Let $\lambda(\omega)|d\omega|$ denote the Poincaré metric of Ω which has constant curvature -4.

THEOREM 1. *If $\Omega\notin O_G$ then, for $\omega\in\Omega$,*

$$k_0(\omega, \omega)\geqq c_\beta(\omega)^2\Big/\left(8\log\frac{\lambda(\omega)}{c_\beta(\omega)}+6\log 2\right). \tag{1.4}$$

We shall reformulate this theorem for Fuchsian groups and then prove it in that form.

If Ω is a Riemann surface such that $c_\beta(\omega)/\lambda(\omega)$ is bounded below then (1.4) implies $k_0(\omega, \omega)\geqq \text{const.}\, c_\beta(\omega)^2$. This assumption holds, in particular, if Ω is a plane domain with uniformly perfect boundary [**5**]. Examples are given by the complement of the Cantor set or the limit set of finitely generated Fuchsian groups.