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On the Stark-Shintani conjecture and cyclotomic Z_p-extensions of class fields over real quadratic fields

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§ 1. Introduction.

Let F be a real quadratic field embedded in the real number field $\boldsymbol{R}$. Let M be a finite abelian extension of F in which exactly one of the two infinite primes of F, corresponding to the prescribed embedding of F into $\boldsymbol{R}$, splits. Let $\mathfrak{f}$ be the conductor of M/F. Denote by $H_F(\mathfrak{f})$ the group consisting of all narrow ray classes of F defined modulo $\mathfrak{f}$. Let G be the subgroup of $H_F(\mathfrak{f})$ corresponding to M by class field theory. Take a totally positive integer ν of F satisfying $\nu+1\in\mathfrak{f}$, and denote by the same letter ν the narrow ray class modulo $\mathfrak{f}$ represented by the principal ideal (ν). For each $c\in H_F(\mathfrak{f})$, set $\zeta_F(s, c)=\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$, where $\mathfrak{a}$ runs over all integral ideals of F belonging to the ray class c. It is known that $\zeta_F(s, c)$ is holomorphic on the whole complex plane except for a simple pole at $s=1$.

The Stark-Shintani ray class invariant $X_{\mathfrak{f}}(c)$ is defined by

$$X_{\mathfrak{f}}(c)=\exp(\zeta_F'(0, c)-\zeta_F'(0, c\nu)) \qquad (c\in H_F(\mathfrak{f})) \tag{1}$$

(see Stark [7] and Shintani [5], the notation $X_{\mathfrak{f}}(c)$ is due to [5]). Obviously, $X_{\mathfrak{f}}(c\nu)=X_{\mathfrak{f}}(c)^{-1}$. In his paper [5], T. Shintani expressed this invariant $X_{\mathfrak{f}}(c)$ as a product of certain special values of the double gamma function of E.W. Barnes. In particular, he proved that $X_{\mathfrak{f}}(c)$ is a positive real number. Put $X_{\mathfrak{f}}(c, G)=\prod_{g\in G} X_{\mathfrak{f}}(cg)$. Then $X_{\mathfrak{f}}(c, G)$ depends only on $c\in H_F(\mathfrak{f})/G$. In [7] and [8], H. M. Stark presented a striking conjecture on the arithmetic nature of the invariant $X_{\mathfrak{f}}(c, G)$. Shintani found it independently and reformulated it in [5] into a more precise form.

CONJECTURE 1. For some positive rational integer m, $X_{\mathfrak{f}}(c, G)^m$ is a unit of $M\,(\forall c\in H_F(\mathfrak{f})/G)$. Moreover, $\{X_{\mathfrak{f}}(c, G)^m\}^{\sigma(c_0)}=X_{\mathfrak{f}}(cc_0, G)^m\ (\forall c_0\in H_F(\mathfrak{f})/G)$, where σ is the Artin isomorphism of $H_F(\mathfrak{f})/G$ onto the Galois group $\mathrm{Gal}(M/F)$.

Shintani introduced in [5] another invariant $Y_{\mathfrak{f}}(c, G)$ to prove Conjecture 1 in some special non-trivial cases (for the definition of $Y_{\mathfrak{f}}(c, G)$, see (18) and (20)