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Leopoldt's conjecture and Reiner's theorem

By Hiroo MIKI and Hirotaka SATO

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§1. Introduction.

Let p be a prime number and let k be a finite algebraic number field. Let k_v be the completion of k with respect to a prime divisor v of k, and let S_k be the set of all prime divisors of k lying over p. Let E_k be the group of units ε of k such that $\varepsilon \in U_v^{(1)}$ for all $v \in S_k$, where $U_v^{(1)}$ is the group of principal units of k_v. Imbed E_k into $\prod_{v \in S_k} U_v^{(1)}$ in the natural way and take the topological closure $\bar{E}_k$ of E_k in $\prod_{v \in S_k} U_v^{(1)}$. Put $\delta_k = \mathrm{rank}_{\boldsymbol{Z}} E_k - \mathrm{rank}_{\boldsymbol{Z}_p} \bar{E}_k$, where $\boldsymbol{Z}$ and $\boldsymbol{Z}_p$ are the rings of integers and p-adic integers respectively. Leopoldt [4] conjectured that $\delta_k = 0$ for any prime number p.

Let K/k be a finite Galois p-extension with Galois group G. In [**7**, Corollary to Theorem 2], we proved the Leopoldt conjecture for (K, p) under certain strong conditions on k and the ramification of K/k. The purpose of the present paper is to give another proof of this theorem by considering the $\boldsymbol{Z}_p[G]$-module structure of the Galois group X_K^* of the composite of all $\boldsymbol{Z}_p$-extensions of K based on Reiner's theorem [**1**, Theorem (74.3)] when K/k is a cyclic extension of degree p (Theorem and its Corollary).

§2. The G-module structure of the Galois group of the composite of $\boldsymbol{Z}_p$-extensions of K.

Let M_k be the maximal p-ramified abelian p-extension of k and let M_k^* be the composite of all $\boldsymbol{Z}_p$-extensions of k. Let L_k and L_k^* be the maximal elementary abelian p-extension of k in M_k and M_k^* respectively. Put $X_k = G(M_k/k)$ and $X_k^* = G(M_k^*/k)$. Then M_K^*/k is a Galois extension and X_K^* becomes a G-module by $\sigma\tau = \tilde{\sigma}\tau\tilde{\sigma}^{-1}$ $(\tau \in X_K^*)$, where σ is a generator of G and $\tilde{\sigma}$ is an extension of σ to M_K^*. From now on, we assume that K/k is unramified at all infinite primes of k if $p=2$. By [**2**, Theorem 3], X_K^* is a free $\boldsymbol{Z}_p$-module of rank $(pr_2+1+\delta_K)$, where $r_2 = r_2(k)$ is the number of complex places of k. Hence by Reiner's theorem [**1**, Theorem (74.3)],