J. Math. Soc. Japan
Vol. 35, No. 4, 1983

A remark on the values of the zeta functions associated with cusp forms

By Yoshitaka MAEDA

(Received July 24, 1982)

Introduction.

For two primitive cusp forms $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)$ and $g(z)=\sum_{n=1}^{\infty}b(n)e(nz)$ ($e(z)=\exp(2\pi iz)$, $z\in\mathfrak{H}$: the upper half complex plane), we define a zeta function by

$$D(s, f, g)=\sum_{n=1}^{\infty}a(n)b(n)n^{-s} \qquad (s\in\boldsymbol{C}),$$

and denote by K the field generated over $\boldsymbol{Q}$ by $a(n)$ and $b(n)$ for all n. If the weight k of f is greater than the weight l of g, Shimura [4] proved that $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)$ belongs to K for an integer m with $(1/2)(k+l-2)<m<k$, where $\langle\ ,\ \rangle$ denotes the normalized Petersson inner product as in [4]. When K is a *CM-field,* namely, a totally imaginary quadratic extension over a totally real field F, we are going to show the divisibility of these special values by a certain polynomial of the Fourier coefficients $a(p)$ and $b(p)$ at prime divisors p of the level of these forms. Roughly speaking, $a(p)-\overline{b(p)}p^{e}$ with a certain integer e depending on k, m and p divides the numerator of $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)$. More precisely, we have

THEOREM 1. *Let χ be the character of f and N the conductor of f. Assume that the character of g is the complex conjugate $\bar{\chi}$ of χ and g has the same conductor N as f. Write M for the conductor of χ. Let A be the set of prime divisors of N satisfying one of the following conditions:*

($\mathrm{C_a}$) *The p-primary part of N is equal to that of M; or,*

($\mathrm{C_b}$) *$p\mid N$, $p^2\nmid N$ and $p\nmid M$.*

Put

$$C=N\times\prod_{p\in A}[a(p)^{\rho}\{a(p)-b(p)^{\rho}p^{k-\delta(p)-m}\}],$$

where

$$\delta(p)=\begin{cases}1 & \textit{if } p \textit{ satisfies Condition } (\mathrm{C_a}),\\ 2 & \textit{if } p \textit{ satisfies Condition } (\mathrm{C_b}),\end{cases}$$