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On the ring of Hilbert modular forms over $\boldsymbol{Z}$

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Introduction.

In the theory of elliptic modular forms, it is known that every modular form whose Fourier coefficients lie in $\boldsymbol{Z}$ is represented as an isobaric polynomial in E_4, E_6, Δ with coefficients in $\boldsymbol{Z}$, where E_k is the normalized Eisenstein series of weight k and $\Delta=2^{-6}\cdot 3^{-3}(E_4^3-E_6^2)$. On the other hand, in his paper [7], J. Igusa gave a minimal set of generators over $\boldsymbol{Z}$ of the graded ring of Siegel modular forms of degree two whose Fourier coefficients lie in $\boldsymbol{Z}$. Also, some related topics and problems on the finite generation of an algebra of modular forms were discussed by W. L. Baily, Jr. in his recent paper [2].

In this paper, we give analogous results for symmetric Hilbert modular forms for the real quadratic fields $\boldsymbol{Q}(\sqrt{2})$ and $\boldsymbol{Q}(\sqrt{5})$. Let $\boldsymbol{K}$ be a real quadratic field and $\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_k$ denote the $\boldsymbol{Z}$-module of symmetric Hilbert modular forms of *even* weight k with rational integral Fourier coefficients and we put $\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})=\bigoplus \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_k$. Denote by G_k the normalized Eisenstein series for the Hilbert modular group $\Gamma_{\boldsymbol{K}}=SL(2, \mathfrak{o}_{\boldsymbol{K}})$. In the case of $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$, we put

$$
\begin{aligned}
H_4&=2^{-6}\cdot 3^{-2}\cdot 11(G_2^2-G_4)\,,\\
H_6&=-2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 5\cdot 7^2 G_2^3+2^{-8}\cdot 3^{-2}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 59 G_2 G_4\\
&\quad -2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2 G_6\,.
\end{aligned}
$$

Our first main result can be stated as follows:

THEOREM 1. *The modular forms G_2, H_4, H_6 all have integral Fourier coefficients, namely, $G_2\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_2$, $H_k\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_k$ ($k=4, 6$). Furthermore, the elements G_2, H_4, H_6 form a minimal set of generators of $\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})$ over $\boldsymbol{Z}$.*

In the case of $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$, we put

$$
\begin{aligned}
J_6&=2^{-5}\cdot 3^{-3}\cdot 5^{-2}\cdot 67(G_2^3-G_6)\,,\\
J_{10}&=2^{-10}\cdot 3^{-5}\cdot 5^{-5}\cdot 7^{-1}(412751 G_{10}-5\cdot 67\cdot 2293 G_2^2 G_6+2^2\cdot 3\cdot 7\cdot 4231 G_2^5)\,,\\
J_{12}&=2^{-2}(J_6^2-G_2 J_{10})\,.
\end{aligned}
$$

The second main theorem is