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On complementary triples in finite groups

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§ 1. Introduction.

All groups considered here are finite. Let H be a subgroup of a group G. We say that H *controls fusion* in H with respect to G if H has the property; 'Two elements of H are conjugate in G if and only if they are conjugate in H.' If H has a normal complement (that is, a normal subgroup N of G with $G=HN$ and $H\cap N=1$) in G, then H controls fusion in H with respect to G. But the converse is false. For example, let S_n be the symmetric group on n letters, where n is greater than 4, and let H be the stabilizer of one point. Then we know that H controls fusion in H with respect to S_n and S_n has no normal subgroups of order n.

What conditions on H guarantee that H has a normal complement? The Brauer-Suzuki theorem answered the question for a Hall subgroup H (see, for example, Theorem 8.22 in [2]). In this paper, we shall give a more general criterion for the existence of a normal complement of a subgroup H in a group G. Before stating our result, we shall introduce the following notation:

Let H be a subgroup of a group G which controls fusion in H with respect to G, and let T, M and L be mappings from $H^{\#}$ to the family of subsets of G, where $H^{\#}=H-\{1\}$. Suppose T, M and L satisfy the following conditions. Then we say (T, M, L) a *complementary triple* of H in G.

(1.1) For every $h\in H^{\#}$,

  (i) $T(h)$ is a subgroup of G with $T(h)^x=T(h^x)$ for $x\in H$,

  (ii) $M(h)=hT(h)$,

  (iii) $L(h)=\bigcup_{g\in G} M(h)^g$,

  (iv) $N_G(M(h))=T(h)C_H(h)$.

(1.2) Whenever $h\in H^{\#}$ and $g\in G$, $M(h)\cap M(h)^g=\emptyset$ or $M(h)$.

(1.3) $(G-\bigcup_{x\in H^{\#}} L(x))\cap N_G(M(h))=T(h)$ for every $h\in H^{\#}$.

(1.4) Whenever h_1 and h_2 are elements of $H^{\#}$ which are not conjugate in G, then $L(h_1)\cap L(h_2)=\emptyset$.

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